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On Generalized Pell Numbers

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The general second order linear recurrence $\{G_n\}$ is defined by constant integers G_0, G_1, A, B and the recursion

$$G_n = A \cdot G_{n-1} - B \cdot G_{n-2}$$

for $n > 1$. If $G_0 = 0$, $G_1 = 1$, $A = -B = 1$ (resp. $G_0 = 1$, $G_1 = 3$, $A = -B = 1$) then the sequence $\{G_n\}$ is called *Fibonacci* (resp. *Lucas*) *sequence*. Another special case of the sequence $\{G_n\}$ is the Pell sequence $\{P_n\}$ defined by $P_0 = 0$, $P_1 = 1$ and

$$P_n = 2P_{n-1} + P_{n-2}$$

for $n > 1$. Generalizing the Pell sequence, we introduce the sequence $\{R_n\}$ defined by constant integers R_0, R_1 and

$$R_n = 2R_{n-1} + R_{n-2}$$

for $n > 1$.

The sequence $\{P_n\}$ and $\{R_n\}$ have been studied by many authors giving a lot of identities (for example H. V. Krishna [7], A. F. Horadam [4]). Furthermore many authors have shown connections between second order recurrences and Pell's equation $x^2 - Dy^2 = N$. In [3] V. E. Hoggatt, Jr. showed that the equation $x^2 - 5y^2 = \pm 4$ has only positive solutions $x = L_n$ and $y = F_n$, where L_n and F_n are the n-th Lucas and Fibonacci numbers respectively. E. M. Cohn [2], I. Adler [1], and V. Thébault [11] bound the

solutions of the Pell's equation $x^2 - 2y^2 = \pm 1$ using second order recurrences and the Pell's numbers. M. J. De Leon [8] proved that if x_0, y_0 is a fundamental solution of $x^2 - 2y^2 = N$, then x_n, y_n is also a solution, for which $x_n + y_n = (x_0 + y_0)\cdot P_{2n+1} + y_0 \cdot P_{2n}$ and $y_n = (x_0 + y_0)\cdot P_{2n} + y_0 \cdot P_{2n-1}$.

Some other results were given by J. P. Jones in [5] and [6], he proved that for Fibonacci and Lucas sequences

$$G_n^2 + A\cdot G_n \cdot G_{n+1} - G_{n+1}^2 = (-1)^n \cdot k$$

for all positive integer n, where $k = -1$ for Fibonacci sequence and $k = -5$ for Lucas sequence. Furthermore he gave polynomials $F(x, y)$ and $L(x, y)$ with integer coefficients of which the positive values are the set of all Fibonacci and Lucas numbers respectively, the variables x and y ranging over the positive integers.

In this paper we point out that Jones' results can be extended also to the Pell numbers. We give a few identities for generalized Pell numbers and express all the solutions of the equation $x^2 - 2y^2 = N$ with the help of generalized Pell numbers. We prove the following theorems:

Theorem 1. *Let* $\{R_n\}$ *be any generalized Pell sequence and let*

$$k = R_0^2 + 2\cdot R_0 \cdot R_1 - R_1^2.$$

Then for every integer $n \geq 0$

(1) $$R_n^2 + 2\cdot R_n \cdot R_{n+1} - R_{n+1}^2 = (-1)^n \cdot k,$$

(2) $$(R_{2n} + R_{2n+1})^2 - 2\cdot R_{2n+1}^2 = k,$$

and for $n > 0$

(3) $$(R_{2n-1} + R_{2n})^2 - 2\cdot R_{2n}^2 = -k.$$

Theorem 2. *Let* N $(\neq 0)$ *be an integer. If the diophantine equation*

(4) $$x^2 - 2y^2 = N$$

has an integer solution, then all the solutions are given by finitely many generalized Pell sequences $\{R_n\}$ *and*

$$(x, y) = (\pm(R_{2n} + R_{2n+1}), \pm R_{2n+1}),$$

where $n = 0, 1, 2, \cdots$. *For these sequences* $0 \le R_1 < 2\sqrt{N}$ *in case* $N > 0$ *and* $0 < R_1 < \sqrt{-9N/2}$ *in case* $N < 0$.

Corollary. *Let* N $(\neq 0)$ *be an integer, then the equations*

$$x^2 - 2y^2 = N$$

and

$$x^2 - 2y^2 = -N$$

are solvable or not at the same time (Note that this assertion follows also from a theorem A. Schinzel and W. Sierpinski [10]).

Theorem 3. *The set* $\{P_n\}$ *of the Pell numbers, defined by* $P_0 = 0$, $P_1 = 1$ *and* $P_n = 2P_{n-1} + P_{n-2}$ *for* $n > 1$, *is identical with the set of the non-negative values of the polynomial*

$$f(x, y) = x(2 - (x^2 + 2xy - y^2)^2),$$

the variables x *and* y *ranging over nin-negative integers.*

Proof of Theorem 1. The equations (2) and (3) follow from equation (1), so we have to prove only the equation (1). The statement of (1) is evident for $n = 0$. If

$$R_i^2 + 2R_iR_{i+1} - R_{i+1}^2 = (-1)^i \cdot k$$

for an integer $i \ge 0$, then

$$R_{i+1}^2 + 2R_{i+1}R_{i+2} - R_{i+2}^2 = R_{i+1}^2 + 2R_{i+1}(2R_{i+1} + R_i) - (2R_{i+1} + R_i)^2$$

$$= -(R_i^2 + 2R_iR_{i+1} - R_{i+1}^2) = -(-1)^i \cdot k = (-1)^{i+1} \cdot k$$

and so (1) holds for every $n \geq 0$ integer.

Proof of Theorem 2. It is enough to look for the (x, y) solutions of the equation (4) for which $x, y \geq 0$, since these yield all the solutions.

If $x^2 - 2y^2 = N$ for integers $x, y \geq 0$, then we can define a generalized Pell sequence by the equations

$$R_{2i+1} = y \quad \text{and} \quad R_{2i} + R_{2i+1} = x$$

or

$$R_{2i} = x - y \quad \text{and} \quad R_{2i+1} = y.$$

This sequence $\{R_n\}$ is defined for each integer index n (n can be positive as well as negative since two consecutive members R_j, R_{j+1} of the sequence $\{R_n\}$ determine the integers R_{j-1} and R_{j+2} by the definition) so we can choose the index i in a way that R_1 is the least non-negative member among the members with odd indices. By Theorem 1, every pair $(x, y) = (R_{2n} + R_{2n+1}, R_{2n+1})$ is a solution of the equation (4) for every integer n, namely in this case $k = N$.

We still have to show that the number of these sequences is finite.

First let $N > 0$. If the integers $x_1, y_1 > 0$ are solutions of (4), then the sequence $\{R_n\}$, generated by x_1 and y_1, has two consecutive members $R_{2i} = x_1 - y_1$ and $R_{2i+1} = y_1$. In this case $R_{2i-1} = R_{2i+1} - 2R_{2i} = 3y_1 - 2x_1$ and $R_{2i-2} = R_{2i} - 2R_{2i-1} = 5x_1 - 7y_1$, by the definition of the sequence $\{R_n\}$; furthermore, by Theorem 1,

$$x_2 = R_{2i-2} + R_{2i-1} + 3x_1 - 4y_1,$$

$$y_2 = R_{2i-1} = 3y_1 - 2x_1$$

are also solutions of the equation (4). But

$$y_2 = 3y_1 - 2x_1 = 3y_1 - 2\sqrt{2y_1^2 + N} < (3 - 2\sqrt{2})y_1 < y_1$$

and

$$x_2 = 3x_1 - 4y_1 = 3x_1 - 4\sqrt{(x_1^2 - N)/2} = 3x_1 - \sqrt{8x_1^2 - 8N} > 0,$$

so we get a sequence $y_1, y_2, \cdots$, for which $y_{i+1} < y_i$ and

$$y_{i+1} = 3y_i - 2\sqrt{2y_i^2 + N} \geq 0$$

if $y_i \geq 2\sqrt{N}$. Therefore there exists an integer index j, for which $0 \leq y_j < 2\sqrt{N}$ and so the sequence $\{R_n\}$ can be generated by the initial values $R_0 = x_j - y_j$ and $R_1 = y_j$, where $0 \leq R_1 < 2\sqrt{N}$.

Thus, in case $N > 0$, all the positive solutions of the equation (4) can be given by sequences $\{R_n\}$, for which $0 \leq R_1 < 2\sqrt{N}$.

Let us study now the case $N < 0$. If $x_1, y_1 > 0$ are solutions of the equation (4), then, as above, $x_2 = 3x_1 - 4y_1$ and $y_2 = 3y_1 - 2x_1$ are also solutions. Continuing the procedure, we show that $x_{i+1} \geq 0$ and $0 < y_{i+1} < y_i$ as long as $y_i \geq \sqrt{-9N/2}$. For if $y_i \geq \sqrt{-9N/2}$, $x_i \geq 0$ and $x_i^2 - 2y_i^2 = N$, then

$$x_i = \sqrt{2y_i^2 + N} \geq \sqrt{-8N},$$

which implies

$$x_{i+1} = 3x_i - 4y_i = 3x_i - 4\sqrt{(x_i^2 - N)/2} \geq 0.$$

Furthermore, by $N < 0$, we get

$$y_{i+1} = 3y_i - 2x_i = 3y_i - 2\sqrt{2y_i^2 + N} > (3 - 2\sqrt{2})y_i > 0$$

and by $y_i \geq \sqrt{-9N/2}$ $(> \sqrt{-N})$ we can easily see that

$$y_{i+1} = 3y_i - 2\sqrt{2y_i^2 + N} < y_i.$$

So in case $N < 0$ the solutions of the equation (4) can be generated by the sequences $\{R_n\}$ for which $0 < R_1 < \sqrt{-9N/2}$ and $x = R_0 + R_1$, $y = R_1$ are solutions.

The *Corollary* immediately follows from Theorem 1 and Theorem 2.

Remark 1. Note that the solutions of the equation $x^2 - 2y^2 = N$ can be given by an explicit formula. It is known that the members of sequence $\{R_n\}$ are

$$R_n = \frac{(R_1 - \beta R_0)\alpha^n - (R_1 - \alpha R_0)\beta^n}{\alpha - \beta}, \tag{5}$$

where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ are the roots of the equation $z^2 - 2z - 1 = 0$ (e.g. see in [9], pp.90-93). Thus, by (5) and Theorem 2, all the solutions of the equation $x^2 - 2y^2 = N$ are

$$\begin{aligned} x &= \pm (R_{2n} + R_{2n+1}) \\ &= \pm \frac{(R_1 + R_0 + \alpha R_1 - \beta R_0)\alpha^{2n} - (R_1 + R_0 + \beta R_1 - \alpha R_0)\beta^{2n}}{\alpha - \beta}, \end{aligned}$$

$$y = \pm R_{2n+1} = \pm \frac{(R_1 - \beta R_0)\alpha^{2n+1} - (R_1 - \alpha R_0)\beta^{2n+1}}{\alpha - \beta}, \tag{6}$$

where $n = 0, 1, 2, \cdots$ and R_1 ranges over the integer values for which $\sqrt{2R_1^2 + N}$ is an integer and for which $0 \le R_1 < 2\sqrt{N}$ in case $N > 0$ and $0 < R_1 < \sqrt{-9N/2}$ in case $N < 0$, furthermore $R_0 = \sqrt{2R_1^2 + N} - R_1$ for every R_1 possible.

Remark 2. One can easily see that the assertions of Theorem 1 and the equation (5) will be valid even if we extend the definition of the sequence $\{R_n\}$ for negative indices n; so $(x_n, y_n) = (R_{2n} + R_{2n+1}, R_{2n+1})$ and (6) give solutions of the equation (4) for $n = -1, -2, \cdots$, too. It decreases the number of the sequences $\{R_n\}$ which generate the solutions.

Example. Let us give all the solutions of the equation

$$x^2 - 2y^2 = 7 \tag{7}$$

as an example. Now $2\sqrt{7} < 6$, therefore if (7) is solvable, then $y = 0$, 1, 2, 3, 4, or 5 is a solution. Replacing the values of y in the

equation (7), we get the solutions $(x, y) = (3, 1)$ and $(5, 3)$. The numbers $x_1 = 3$ and $y_1 = 1$ generate the sequence $\{R_n\}$ with $R_0 = x_1 - y_1 = 2$, $R_1 = y_1 = 1$ and the members of the sequence are $\cdots$, $R_{-4} = 46$, $R_{-3} = -19$, $R_{-2} = 8$, $R_{-1} = -3$, $R_0 = 2$, $R_1 = 1$, $R_2 = 4$, $R_3 = 9$, $R_4 = 22$, $R_5 = 53$, $\cdots$. So $(x, y) = \cdots, (\pm 27, \pm 19), (\pm 5, \pm 3), (\pm 3, \pm 1), (\pm 13, \pm 9), (\pm 75, \pm 53), \cdots$ are the solutions of (7). These are all the solutions of the equation (7) because $y = 3$ occurs among them and so $x_1 = 5$, $y_1 = 3$ do not generate new solutions.

In our case

$$\alpha - \beta = (1 + \sqrt{2}) - (1 - \sqrt{2}) = 2\sqrt{2},$$

$$R_1 + R_0 + \alpha R_1 - \beta R_0 = 2 + 3\sqrt{2},$$

$$R_1 + R_0 + \beta R_1 - \alpha R_0 = 2 - 3\sqrt{2},$$

$$R_1 - \beta R_0 = -1 + 2\sqrt{2},$$

and

$$R_1 - \alpha R_0 = -1 - 2\sqrt{2},$$

therefore all the solutions of the equation $x^2 - 2y^2 = 7$ are

$$x = \pm \frac{(2 + 3\sqrt{2}) \cdot (1 + \sqrt{2})^{2n} - (2 - 3\sqrt{2}) \cdot (1 - \sqrt{2})^{2n}}{2\sqrt{2}},$$

$$y = \pm \frac{(-1 + 2\sqrt{2}) \cdot (1 + \sqrt{2})^{2n+1} - (-1 - 2\sqrt{2}) \cdot (1 - \sqrt{2})^{2n+1}}{2\sqrt{2}},$$

where $n = 0, \pm 1, \pm 2, \cdots$.

Proof of Theorem 3. Let us solve the equations

$$p^2 - 2q^2 = 1 \tag{8}$$

and

$$p^2 - 2q^2 = -1. \tag{9}$$

The solution of (8) if $0 \le q < 2$ is $p_1 = 1$, $q_1 = 0$ and so all the

solutions of (8) are given by the sequence $\{R_n\}$ with $R_0 = p_1 - q_1 = 1$ and $R_1 = q_1 = 0$. This sequence $\{R_n\}$ is identical with the Pell sequence $\{P_n\}$ since $R_2 = 2R_1 + R_0 = 1$, and $R_n = P_{n-1}$ for every integer n. It implies, by Theorem 2, that the non-negative solutions of (8) are $(p, q) = (P_{2n-1} + P_{2n}, P_{2n})$, where $n = 0, 1, 2, \cdots$ are $P_{-1} = 1$.

The solution of the equation (9), if $0 < q < \sqrt{9/2}$, is $(p, q) = (1, 1)$ and one can see as above that all the non-negative solutions of the equation (9) are $(p, q) = (P_{2n} + P_{2n+1}, P_{2n+1})$, where $n = 0, 1, 2, \cdots$.

Thus for $p, q \geq 0$

$$|p^2 - 2q^2| = 1$$

if and only if q is a Pell number and p is the sum of two Pell numbers. From this it follows that for $x, y \geq 0$

$$|x^2 + 2xy - y^2| = |(x + y)^2 - 2y^2| = 1$$

if and only if x and y are consecutive Pell numbers. But for integers $x^2 + 2xy - y^2 = 0$ if and only if $x = y = 0$, therefore the values

$$r = 2 - (x^2 + 2xy - y^2)^2$$

for non-negative integers x, y are $r = 2$ if and only if $x = y = 0$, $r = 1$ if and only if x and y are any consecutive Pell numbers, and $r < 0$ in other cases. From this it follows the statement of the Theorem 3.

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