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On Fixed Points of Densifying Mappings

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The concept of the measure of non-compactness of a bounded set in a metric space was introduced by Kuratowski [2].

Definition 1. Let A be a bounded set in a metric space (X, d) . Then the *measure of non-compactness* of A , written as $\alpha(A)$, is the infimum of all $\varepsilon > 0$ such that A admits a finite covering by sets with diameter less than ε .

Definition 2 (Furi and Vignoli [1]). A mapping T of a metric space (X, d) into itself is said to be *densifying* if for every bounded subset A of X , with $\alpha(A) > 0$

$$\alpha(T(A)) < \alpha(A).$$

Definition 3. Let X be a set and let F be a real-valued lower semi-continuous function of $X \times X$ into $[0, \infty)$. Then two mappings S and T of X into itself are said to be *jointly F -contractive* if for all x, y in X

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$$F(Sx, TSy) < F(x, Sy), x \neq Sy$$

and

$$F(Tx, STy) < F(x, Ty), x \neq Ty.$$

We now prove the following theorem

Theorem. *Let S and T be two continuous densifying and jointly F -contractive mappings of a complete metric space (X, d) into itself. If for some x_0 in X , the sequence $\{x_n\}$ defined by $Sx_{2n} = x_{2n+1}$, $Tx_{2n+1} = x_{2n+2}$ for $n = 0, 1, 2, \dots$ is bounded, then either S or T has a fixed point z . Further, if z is a common fixed point of S and T , then it is the unique fixed point of S and T .*

Proof. Consider the set

$$A = \bigcup_{n=0}^{\infty} \{x_{2n}\}.$$

Then

$$TS(A) = \bigcup_{n=0}^{\infty} \{x_{2n+2}\},$$

where obviously $TS(A) \subset A$. The continuity of S and T now implies that

$$TS(\overline{A}) \subset \overline{TS(A)}.$$

We will now prove that $\overline{A}$ is compact. To do this it is sufficient to show that $\alpha(A) = 0$ since X is complete. Suppose that $\alpha(A) > 0$.

Then $A = \{x_0\} \cup TS(A)$ implies that

$$\begin{aligned} \alpha(A) &= \alpha(\{x_0\} \cup TS(A)) \\ &= \max\{\alpha(x_0), \alpha(TS(A))\} \\ &= \alpha(TS(A)) \\ &< \alpha(A), \end{aligned}$$

giving a contradiction. It follows that $\overline{A}$ is compact.

Now define a real-valued function f on $\bar{A}$ by

$$f(x) = F(x, Sx),$$

for all x in $\bar{A}$. Then F is a lower semi-continuous function on the compact metric space $\bar{A}$ and so there exists z in $\bar{A}$ such that

$$f(z) = \inf\{f(x) : x \in \bar{A}\}.$$

Suppose that $Sz \neq z$, $TSz \neq Sz$. Then

$$\begin{aligned} f(TSz) &= F(TSz, STSz) \\ &< F(Sz, TSz) \\ &< F(z, Sz) \\ &= f(z), \end{aligned}$$

contradicting the definition of z . It follows that either $Sz = z$, in which case z is a fixed point of S , or $TSz = Sz$, in which case $z' = Sz$ is a fixed point of T .

We will now show that if z is a common fixed point of S and T then it is the unique fixed point of S and T . To do this, let us suppose that S has a second distinct fixed point z' . Then $Sz' \neq z$ and so

$$\begin{aligned} F(z', z) &= F(Sz', TSz) \\ &< F(z', Sz) \\ &= F(z', z), \end{aligned}$$

giving a contradiction. It follows that z is the unique fixed point of S . Similarly z is the unique fixed point of T . This completes the proof of the theorem.

To see that S and T do not necessarily have a common fixed point let $X = \{x, y\}$ with $F = d$, the discrete metric. Define mappings S

and T on X by

$$Sx = x, Sy = y, Tx = y, Ty = x.$$

Then S and T are jointly F -contractive but S has two fixed points and T has no fixed points.

The following corollary follows immediately from the theorem on putting $S = T$.

Corollary 1. *Let T be a continuous densifying mapping of a complete metric space (X, d) into itself and let F be a real-valued lower semi-continuous function of $X \times X$ into $[0, \infty)$ such that for all x, y in X*

$$F(Tx, T^2y) < F(x, Ty), x \neq Ty.$$

If for some x_0 in X the sequence $\{x_n\}$ defined by $Tx_n = x_{n+1}$ for $n = 0, 1, 2, \dots$ is bounded, then T has a unique fixed point z .

Corollary 2 (Furi and Vignoli [1]). *Let T be a continuous densifying mapping of a complete metric space (X, d) into itself and let F be a real-valued lower semi-continuous function of $X \times X$ into $[0, \infty)$ such that for all x, y in X*

$$F(Tx, Ty) < F(x, y), x \neq y.$$

If for some x_0 in X the sequence $\{x_n\}$ defined by $Tx_n = x_{n+1}$ for $n = 0, 1, 2, \dots$ is bounded, then T has a unique fixed point z .

Proof. Since

$$F(Tx, Ty) < F(x, y), x \neq y$$

it follows that

$$F(Tx, T^2y) < F(x, Ty), x \neq Ty.$$

The result now follows from Corollary 1.

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