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A Fixed Point Theorem for Commuting Mappings

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1. In a recent paper, Jungck [2] investigates an interdependence between commuting mappings and fixed points and proves the following:

Theorem A. *Let f be a continuous mapping of a complete metric space (X, d) into itself. Then f has a fixed point in X if and only if there exist an $\alpha \in (0, 1)$ and a mapping $g: X \to X$ which commutes with f and satisfies*

(1.1) $g(X) \subset f(X)$ *and* $d(g(x), g(y)) \leq \alpha d(f(x), f(y))$ *for all* $x, y \in X$.

Indeed, f and g have a unique common fixed point if and only if (1.1) *holds for some* $\alpha \in (0, 1)$.

The above result was generalized by Singh [3] who proved the following:

Theorem B. *Let two continuous and commuting mappings S and T from a complete metric space (X, d) into itself satisfy the following conditions*

(1.2) $S(X) \subset T(X)$

(1.3) $d(Sx, Sy) \leq a[d(Tx, Sx) + d(Ty, Sy)] + b[d(Tx, Sy) + d(Ty, Sx)] + cd(Tx, Ty)$

for all x, y in X where a, b, c are non-negative reals satisfying $0 < 2a + 2b + c < 1$. *Then S and T have a unique common fixed point.*

In the proof of Theorem A, use is made of the continuity of g, which is a consequence of (1.1), and in Theorem B, the continuity of S is assumed and used in the proof. We shall prove in this paper a stronger theorem which generalizes both the above results and shows that use need not be made of these continuity arguments.

2. The main result is the following.

Theorem. *Let T be a continuous mapping of a complete metric space (X, d) into itself. Then T has a fixed point in X if and only if there exist a real number $q \in (0, 1)$ and a mapping $S:X \to X$ which commutes with T satisfying*

(2.1) $S(X) \subset T(X)$

(2.2) $d(Sx, Sy) \leq q \cdot \max \{ d(Tx, Sx), d(Ty, Sy), d(Tx, Sy), d(Ty, Sx), d(Tx, Ty)\}$

for all x, y in X.

Indeed commuting mappings T and S have a unique common fixed point if (2.2) *holds for some $q \in (0, 1)$.*

Proof. Let x_0 be an arbitrary point of X, and let x_1 be such that $Tx_1 = Sx_0$. In general, choose x_n such that $Tx_n = Sx_{n-1}$. This can be done because of (2.1). We now show that the sequence $\{Tx_n\}$ is Cauchy.

$$\begin{aligned}(2.3)\quad d(Tx_i, Tx_j) &= d(Sx_{i-1}, Sx_{j-1}) \\ &\leq q \cdot \max \{ d(Tx_{i-1}, Sx_{i-1}), d(Tx_{j-1}, Sx_{j-1}), \\ &\qquad d(Tx_{i-1}, Sx_{j-1}), d(Tx_{j-1}, Sx_{i-1}), d(Tx_{i-1}, Tx_{j-1})\}\end{aligned}$$

where $1 \leq i, j \leq n$. Denote the set $\{Tx_0, Tx_1, \cdots, Tx_n\}$ by $O(x_0, n)$ and let the diameter of $O(x_0, n)$ be d. Then from (2.3) it follows that $d(Tx_i, Tx_j) \leq qd < d$ where $1 \leq i, j \leq n$. This means that there is a

positive integer $k \le n$ such that $d(Tx_0, Tx_k) = d$. So we have

$$\begin{aligned} d(Tx_0, Tx_k) &\le d(Tx_0, Tx_1) + d(Tx_1, Tx_k) \\ &\le d(Tx_0, Tx_1) + qd \\ &= d(Tx_0, Tx_1) + qd(Tx_0, Tx_k), \end{aligned}$$

which implies $d(Tx_0, Tx_k) \le \frac{1}{1-q} d(Tx_0, Tx_1)$. Hence $\delta[O(x_0, n)]$, the diameter of $O(x_0, n)$, is not greater than $\frac{1}{1-q} d(Tx_0, Tx_1)$ for every $n \ge 1$. Now let $m > n$. Then we have $d(Tx_n, Tx_m) \le q\delta[O(x_{n-1}, m - n + 1)]$, where $O(x_{n-1}, m - n + 1) = \{Tx_{n-1}, Tx_n, \cdots, Tx_m\}$. As before there exists a positive integer k_1, $n \le k_1 \le m$, such that

$$d(Tx_{n-1}, Tx_{k_1}) = \delta[O(x_{n-1}, m - n + 1)].$$

On the other hand, since

$$\begin{aligned} d(Tx_{n-1}, Tx_{k_1}) &\le q\delta[O(x_{n-2}, k_1 - n + 2)] \\ &\le q\delta[O(x_{n-2}, m - n + 2)], \end{aligned}$$

we obtain

$$\begin{aligned} d(Tx_n, Tx_m) &\le q\delta[O(x_{n-1}, m - n + 1)] \\ &\le q^2\delta[O(x_{n-2}, m - n + 2)] \\ &\quad \cdots\cdots\cdots \\ &\le q^{n-1}\delta[O(x_1, m - 1)] \\ &\le q^n\delta[O(x_0, m)] \\ &\le \frac{q^n}{1-q} d(Tx_0, Tx_1) \end{aligned}$$

Since $q < 1$, letting $n \to \infty$ we see that $\{Tx_n\}$ is a Cauchy sequence. Since X is complete, $\{Tx_n\}$ converges to some $t \in X$, and hence $\{Sx_n\}$ also converges to t. So the continity of T gives $\{TTx_n\} \to Tt$ and $\{TSx_n\} \to Tt$. Since $ST = TS$, we also have $\{STx_n\} \to Tt$. We now prove that $\{STx_n\}$ converges to St. We have

$$d(STx_n, St) \le q.\max\{d(TTx_n, STx_n), d(Tt, St), d(TTx_n, St),$$

$$d(Tt, STx_n), d(TTx_n, Tt)\}$$

$$\leq q[d(TTx_n, Tt) + d(Tt, STx_n) + d(STx_n, St)],$$

and so we obtain

$$d(STx_n, St) \leq \frac{q}{1-q}[d(TTx_n, Tt) + d(Tt, STx_n)],$$

which shows that $\{STx_n\} \to St$. Therefore $St = Tt$. Consequently, $T(Tt) = T(St) = TS(t) = ST(t) = S(Tt) = S(St)$, and hence we have

$$d(St, S(St)) \leq q \cdot \max\{d(Tt, St), d(TSt, SSt), d(Tt, SSt),$$

$$d(TSt, St), d(Tt, TSt)\}$$

$$= qd(St, SSt),$$

which implies $d(St, SSt) = 0$ or $SSt = St$. Hence $T(St) = S(St) = St$. Thus St $(= Tt)$ is a common fixed point of S and T. If z is any other common fixed point of S and T, then

$$d(z, St) = d(Sz, St)$$

$$\leq q \cdot \max\{d(Tz, Sz), d(Tt, St), d(Tz, St), d(Tt, Sz), d(Tz, Tt)\}$$

$$= qd(z, St),$$

which yields $z = St$. Hence the common fixed point is unique. This proves the sufficiency. The necessity can be proved readily: Suppose $T(a) = a$ for some a in X. Define the constant mapping $S: X \to X$ by letting $S(x) = a$ for all $x \in X$. Then $S(X) = \{a\} \subset T(X)$ and $ST = TS$; condition (2.2) holds trivially since $d(Sx, Sy) = 0$ for all $x, y \in X$. This completes the proof.

Corollary 1. *Let T and S be commuting mappings of a complete metric space (X, d) into itself. Suppose that T is continuous and $S(X) \subset T(X)$. If there exist a $q \in (0, 1)$ and a positive integer k such that*

$$d(S^k x, S^k y) \leq q \cdot \max\{d(Tx, S^k x), d(Ty, S^k y), d(Tx, S^k y),$$

$$d(Ty, S^k x), d(Tx, Ty)\}$$

for all $x, y \in X$, *then* T *and* S *have a unique common fixed point.*

Proof. We have $S^k(X) \subset S(X) \subset T(X)$ and S^k commutes with T. So the theorem ensures the existence of a unique common fixed point $a \in X$ of S^k and T. Now since

$$S(a) = S(Ta) = T(Sa)$$

and

$$S(a) = S(Ta) = S(S^k a) = S^k(Sa),$$

it follows that $S(a)$ is a common fixed point of S^k and T. Thus the uniqueness of a implies $a = Sa$. Therefore $a = Sa = Ta$.

Corollary 2. *Let* (X, d) *be a complete metric space,* k *a real number greater than* 1, *and* n *a positive integer. If* S *is a mapping of* X onto *itself such that* S^{n+1} *is continuous and*

(2.4) $$kd(x, y) \leq \max\{d(S^n x, x), d(S^n y, y), d(S^n x, y), d(S^n y, x), d(S^n x, S^n y)\}$$

for all $x, y \in X$, *then* S *has a unique fixed point.*

Proof. Since $S(X) = X = S^{n+1}(X)$, the inequality (2.4) shows that the theorem is applicable to S and S^{n+1}. Thus we have the desired conclusion.

Remark. Let S be a mapping of a complete metric space (X, d) into itself. A theorem of Ćirić in [1] shows that if there exists a $q \in (0, 1)$ such that

(2.5) $$d(Sx, Sy) \leq q \cdot \max\{d(x, Sx), d(y, Sy), d(x, Sy), d(y, Sx), d(x, y)\}$$

for all $x, y \in X$, then S has a unique fixed point. This follows also from our theorem. In fact, it is clear that such an S commutes with the

identity mapping T on X, $S(X) \subset T(X)$, and that T is continuous. Hence (2.5) shows that our theorem is applicable to S and T. Thus S and T have a unique common fixed point $a \in X$. Since every point of X is fixed under T, the mapping S cannot have any other fixed point, for this would contradict the uniqueness of a. Therefore a is the unique fixed point of S.

The following example illustrates the theorem:

Example. Let $X = \{1, 2, 3\}$, and define the metric d on X by letting $d(1, 2) = 2$, $d(2, 3) = 3$, and $d(1, 3) = 4$. Consider the mappings T and S defined by $T(1) = 1$, $T(2) = T(3) = 3$, and $S(x) = 1$ for all x in X. Obviously T and S satisfy all the conditions of our theorem, and the unique common fixed point is 1. Note that T does not satisfy Ćirić's condition (2.5). In fact, if it did, we have

$$\begin{aligned} 4 = d(1, 3) &= d(T(1), T(2)) \\ &\leq q \cdot \max\{d(1, 1), d(2, 3), d(1, 3), d(2, 1), d(1, 2)\} \\ &= 4q < 4, \end{aligned}$$

a contradiction.

Moreover this example shows that the fixed point of the mapping T in our theorem is not necessarily unique.

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