Kobe University Repository : Kernel

PDF issue: 2026-11-04

On the Bi-Ideal Semigroup of a Semigroup

Lajos, Sándor

(Citation)
Mathematics Seminar Notes,6(3):419-424

(Issue Date)
1978

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/E0001429

(URL)
https://hdl.handle.net/20.500.14094/E0001429

KOBE UNIVERSITY

MATHEMATICS SEMINAR NOTES
Vol. 6 (1978) September 1978

On the Bi-Ideal Semigroup of a Semigroup

By Sándor LAJOS

Let S be a semigroup. A subsemigroup A of S is said to be a *bi-ideal* or (1, 1)-*ideal* of S if $ASA \subset A$. In 1961 the author [3] proved that a non-empty subset A of S is a bi-ideal of S if and only if it is a left [right] ideal of a right [left] ideal of S. Furthermore, the product of two bi-ideals of a semigroup S is again a bi-ideal of S. Thus the collection of all bi-ideals of a semigroup S is a semigroup with respect to set multiplication. This semigroup will be denoted by $B(S)$. It is natural to inquire what classes of semigroups can be characterized by properties of the bi-ideal semigroup. In this note some results will be given in this direction.

For the notation and terminology not defined here, the reader is referred to [1], [2], and [11].

In what follows we need the following two lemmas.

Lemma 1 (Luh [9]). *A semigroup S is regular if and only if*

$$QSQ = Q \tag{1}$$

holds for every quasi-ideal Q of S.

Lemma 2 (Lajos [4]). *A semigroup S is a semilattice of groups if and only if the condition*

(2) $$A \cap B = AB$$

holds for every couple of bi-ideals of S.

Theorem 1. *A semigroup* S *is a semilattice of groups if and only if the bi-ideal semigroup of* S *is a regular monoid.*

Proof. If S is a semilattice of groups, then Lemma 2 implies

(3) $$L \cap R = RL$$

for every left ideal L and every right ideal R of S, whence it follows that S is regular. Hence, for any bi-ideal A of S,

(4) $$ASA = A,$$

and $B(S)$ is regular. On the other hand, Lemma 2 implies $A = AS = SA$ for every bi-ideal A of S, i.e. S is the identity element of $B(S)$.

Conversely, if S is a semigroup whose bi-ideal semigroup $B(S)$ is a regular monoid, then Lemma 1 implies that S is regular. We show that the identity of $B(S)$ is S. If E is the identity of $B(S)$, then $S = ESE \subset E$, whence $E = S$. Thus $BS = SB = B$ for every (1, 1)-ideal B of S. Hence B is a two-sided ideal of S. It follows that S is a regular duo semigroup which is a semilattice of groups.

Theorem 2. *A semigroup* S *is a semilattice of groups if and only if the bi-ideal semigroup* $B(S)$ *of* S *is a distributive lattice with respect to set product and set-theoretical union.*

Proof. If S is a semilattice of groups, then Lemma 2 implies that every bi-ideal of S is globally idempotent and $AB = BA$ for any two bi-ideals of S. Thus $B(S)$ is a multiplicative semilattice. Lemma 2 implies that $B = BS = SB$ for every bi-ideal B, i.e. every bi-ideal of S

is a two-sided ideal of S. Thus $B(S)$ is the collection of all (two-sided) ideals of S, which is a semilattice with respect to set-theoretical union. Also the absorption laws and the distributive laws hold in $B(S)$. Therefore $B(S)$ is a distributive lattice.

Conversely, if a semigroup S has a distributive lattice as the bi-ideal semigroup, then $B(S)$ is a commutative idempotent monoid with respect to set product. Then Theorem 1 ensures that S is a semilattice of groups. Theorem 2 is completely proved.

Lemma 3 (Lajos [5]). *A semigroup S is a semilattice of left groups if and only if the condition*

$$B \cap L = BL \tag{5}$$

holds for every bi-ideal B and every left ideal L of S.

Theorem 3. *A semigroup S is a semilattice of left groups if and only if the bi-ideal semigroup $B(S)$ of S is regular and S is a right identity of $B(S)$.*

Proof. If S is a semigroup which is a semilattice of left groups, then Lemma 3 implies (3) for every left ideal L and every right ideal R of S, i.e. S is regular. Then, by Lemma 1, it follows that $B(S)$ is regular. Finally, (5) implies $B = BS$ for any bi-ideal B of S, that is, S is a right identity of $B(S)$, indeed.

Conversely, if $B(S)$ is regular and S is a right identity of $B(S)$, then Lemma 1 implies that S is regular (because $BSB = B$ holds for every bi-ideal B of S). On the other hand, we have $BS = B$ for every bi-ideal B, whence B is a right ideal of S. Thus S is a regular

left duo semigroup which is a semilattice of left groups.

Lemma 4 (Saitô [10]). *A semigroup S is a semilattice of left simple semigroups if and only if the condition*

$$L_1 \cap L_2 = L_1 L_2 \tag{6}$$

holds for every couple of left ideals of S.

Theorem 4. *A semigroup S is a semilattice of left simple semigroups if and only if the collection of all left ideals of S is a distributive lattice with respect to set product and set union.*

Proof. If S is a semilattice of left simple semigroups, then Lemma 4 implies $L_1 L_2 = L_2 L_1$ for any two left ideals and every left ideal of S is globally idempotent. Thus the set $L(S)$ of all left ideals of S is a multiplicative semilattice. Lemma 4 implies $L = LS$, i.e. every left ideal is a right ideal of S, whence S is left duo. Thus $L(S)$ is the collection of all (two-sided) ideals of S. This is a semilattice with respect to set union and the absorption laws as well as the distributive lwas hold. Therefore $L(S)$ is a distributive lattice, indeed.

Concersely, if a semigroup S has the property that the set of left ideals is a distributive lattice, then $L(S)$ is a multiplicative semilattice. Hence it follows that

$$L_1 \cap L_2 = (L_1 \cap L_2)^2 \subset L_1 L_2 \tag{7}$$

for every couple of left ideals. On the other hand, $L_1 L_2 = L_2 L_1 \subset L_1 \cap L_2$, and we get (6) for any two left ideals of S. Thus, by Lemma 4, S is a semilattice of left simple semigroups.

A semigroup S is called a *homogroup* if it has a subgroup which is a

two-sided ideal of S. For instance, every semigroup with zero element is a homogroup.

Theorem 5. *A semigroup S is a homogroup if and only if the bi-ideal semigroup of S has a zero element.*

Proof. Let S be a homogroup with the group-ideal G. Let B be a bi-ideal of S. Then the product BG is a right ideal of S, and $BG \subset G$. Hence it follows that $BG = G$, because a group has no proper right ideal. Similarly, we get $GB = G$ and G is the zero element of $B(S)$.

Conversely, if S is a semigroup whose bi-ideal semigroup $B(S)$ has a zero element, then

$$SZ = ZS = Z, \tag{8}$$

where Z is the zero of $B(S)$. Hnece Z is a two-sided ideal of S. For any element z of Z, the product Zz is a bi-ideal of S. Thus we have $Z(Zz) = Zz = Z$, because the set product is associative. Similarly. we get $zZ = Z$ for any element z of Z. Therefore Z is a subgroup of S, and S is a homogroup.

Theorem 6. *For a semigroup S the bi-ideal semigroup $B(S)$ is a monoid if and only if* (i) *every bi-ideal of S is a two-sided ideal of S, and* (ii) *every two-sided ideal I of S is complete (i.e. $IS = SI = I$).*

Proof. First, let S be a semigroup having the properties (i), (ii). Then $B(S)$ is the multiplicative monoid of all two-sided ideals of S with identity S.

Secondly, if the bi-ideals of a semigroup S form a monoid with identity A, then $S = ASA \subset A$, whence $A = S$. Thus $BS = SB = B$ for every

bi-ideal B of S, i.e. every bi-ideal is a complete ideal of S.

Theorem 6 is completely proved.

References

[1] A. H. Clifford and G. B. Preston: The algebraic theory of semigroups, vol. I, 2nd edition, Amer. Math. Soc., Providence R. I., 1964.

[2] J. M. Howie: An introduction to semigroup theory, Academic Press, London-New York-San Francisco, 1976.

[3] S. Lajos: Genera-ized ideals in semigroups, Acta Sci. Math., 22 (1961), 217-222.

[4] S. Lajos: On semilattices of groups, Proc. Japan Acad., 45 (1969), 383-384.

[5] S. Lajos: Notes on regular semigroups IV, Math. Balkan., 2 (1972), 116-117.

[6] S. Lajos: A note on semilattices of groups, Acta Sci. Math., 33 (1972), 315-317.

[7] S. Lajos: A characterization of Cliffordian semigroups, Proc. Japan Acad., 52 (1976), 496-497.

[8] S. Lajos and G. Szász: Generalized regularity in semigroups, K. Marx Univ. Economics, Dept. Math., Budapest, 1975.

[9] J. Luh: A characterization of regular rings, Proc. Japan Acad., 39 (1963), 741-742.

[10] T. Saitô: On semigroups which are semilattices of left simple semigroups, Math. Japonicae, 18 (1973), 95-97.

[11] G. Szász: Introduction to lattice theory, Academic Press, Budapest-New-York, 1963.

[12] G. Thierrin: Sur les homo-groups, C. R. Acad. Sci. Paris, 234 (1952), 1519-1521.

H-1194 Budapest
Pincér u.8.
Hungary