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Lattice Ordered Autometrized Algebras

By B. V. SUBBA RAO

Introduction. Prabhakara Rao [3] recently studied seminormal autometrized algebras in an attempt to obtain a unified theory of autometrized algebras. He has studied the ideal theory of seminormal autometrized algebras. Our aim now is to show that the notion of an ideal in an autometrized algebra can be successfully exploited to obtain the structure of autometrized algebras. We show this by studying an extensive subclass, namely the class of representable autometrized algebras, of seminormal autometrized algebras. This class includes DRℓ-semigroups [6], and consequently, Brouwerian algebras, Boolean algebras and ℓ-groups as special cases. Also this class is wider than the class of DRℓ-semigroups (see Example 2.3.1 below). We give a necessary and sufficient condition for representing any member of this class as a subdirect product of totally ordered autometrized algebras (see Theorem 2.12 below). This theorem generalizes a corresponding result for DRℓ-semigroups [8], and consequently it generalizes a well known theorem of Stone that any Boolean algebra is a subdirect product of two element Boolean algebras and also Clifford's theorem (Theorem 22 of [1]) that any commutative ℓ-group is a subdirect product of totally ordered groups.

This paper is divided into two sections. In § 1, we characterize lat-

tice ordered groups and DRℓ-semigroups with least element among the class of autometrized algebras. In §2, we introduce and study representable autometrized algebras.

All the DRℓ-semigroups and ℓ-groups considered in this paper are commutative. We freely make use of the definitions and results regarding ℓ-groups and DRℓ-semigroups from [1] and [6] respectively.

§ 1. We recall now the definitions of an autometrized algebra [4] and of a semiregular autometrized algebra [4].

Definition 1.1. A system $A = (A, +, \leq, *)$ is called an *autometrized algebra* if and only if

A1) $(A, +, \leq)$ is a commutative, partially ordered semigroup with additive identity 0, i.e., $(A, +)$ is a commutative semigroup with additive identity 0 and $(A, \leq)$ is a partially ordered set such that $a \leq b$ implies $a + x \leq b + x$ for any x in A.

A2) $*$ is a metric operation on A, i.e., $*$ is a mapping from $A \times A$ to A such that

M1) $a * b \geq 0$ for all a, b in A and $a * b = 0$ if and only if $a = b$

M2) $a * b = b * a$ for all a, b in A

M3) $a * c \leq a * b + b * c$ for all a, b, c in A.

Definition 1.2. An autometrized algebra $A = (A, +, \leq, *)$ is called a *semiregular autometrized algebra* if and only if $a \geq 0$ implies $a * 0 = a$.

We now give below a characterization of lattice ordered groups among the class of autometrized algebras.

Theorem 1.3. *Let* $A = (A, +, \leq, *)$ *be an autometrized algebra. Then* A *is an* ℓ*-group with* $a * b = |a - b|$ *if and only if* A *satisfies the following*:

(1.3.1) A *is a semiregular autometrized algebra.*

(1.3.2) $a \leq a * 0$ *for all* a *in* A.

(1.3.3) *For every* a *in* A, *the mapping* $x \mapsto a + x$, *is an isometry from* A *onto* A.

(1.3.4) $2a \geq 0$ *implies* $a \geq 0$, *for each* a *in* A.

(1.3.5) *For every* a *in* A, *the equation* $a + (a * 0) = 2x$ *is solvable in* A.

Proof. Assume that $A = (A, +, \leq, *)$ satisfies the condition (1.3.1) through (1.3.5). Clearly $A = (A, +, \leq)$ is a commutative partially ordered semigroup with additive identity 0. For each $a \in A$, let f_a denote the mapping $x \mapsto a + x$. Let $a \in A$. Since $0 \in A$ and f_a is an isometry onto A (by 1.3.3), there exists a unique x such that $f_a(x) = 0$ so that $a + x = 0 = x + a$. We denote this unique x by $-a$. Thus $-a$ is the additive inverse of a. Hence $(A, +, \leq)$ is a commutative po-group. In order to show that $(A, +, \leq)$ is an ℓ-group, it is enough, in view of Lemma 1 ([1], page 292) to show that for all $a \in A$, the least upper bound $a \cup 0 = a^+$ of a and 0 exists in A. Let $a \in A$. Then by (1.3.5), the equation $a + a*0 = 2x$ is solvable in A. Suppose $a + a*0 = 2x$ and $a + a*0 = 2y$ for x, y in A. Then $2x - 2y = 0$. This implies $2(x - y) = 0$ so that $2(x - y) \geq 0$ and $2(y - x) \geq 0$. Hence $x - y \geq 0$ and $y - x \geq 0$ by (1.3.4). This implies $x \geq y$ and $y \geq x$. Hence $x = y$. Thus, the equation $a + a*0 = 2x$ is uniquely solvable in

A. Let x_a be the unique solution of $a + a*0 = 2x$ so that $a + a*0 = 2x_a$. We now prove that x_a is the least upper bound of a and 0. $2x_a = a + a*0 \geq a + a$ (by 1.3.2) $= 2a$. This implies $2(x_a - a) \geq 0$ which implies $x_a - a \geq 0$ (by 1.3.4). Hence $x_a \geq a$. Also, $2x_a = a + a*0 = a + (a - a) * (0 - a)$ (by 1.3.3) $= a + 0*(-a) \geq a + (-a)$ (by 1.3.2) $= 0$. Hence $x_a \geq 0$ (by 1.3.4). Now, let $a \leq y$ and $0 \leq y$. Then $y - a \geq 0$ and $y \geq 0$. So, $2x_a = a + a*0 \leq a + a*y + y*0$ (by M3) $\leq a + (a - a) * (y - a) + y*0$ (by 1.3.3) $= a + 0*(y - a) + y*0 = a + (y - a) + y$ (by 1.3.1) $= 2y$. Hence $2(y - x_a) \geq 0$. This implies $y - x_a \geq 0$ (by 1.3.4) which implies that $y \geq x_a$. Thus x_a is the least upper bound of a and 0. Thus $a \cup 0 = a^+$ exists for all a in A. Hence $A = (A, +, \leq)$ is a commutative ℓ-group. Now, let $a \in A$. We have $a \leq a*0$ and $-a \leq (-a)*0 = 0*a$ (by 1.3.3) $= a*0$. Let $a \leq b$ and $-a \leq b$. Then $2a \leq a + b$ and $0 \leq a + b$. This implies $(2a) \cup 0 \leq a + b$, which implies $2(a \cup 0) \leq a + b$. Hence $a + a*0 \leq a + b$. This implies $a*0 \leq b$. Thus $a*0$ is the least upper bound of a and $-a$. Hence $a*0 = a \cup (-a) = |a|$, the absolute value of a in the ℓ-group A. So, $a*b = (a-b) * (b-b) = (a-b)*0 = |a-b|$. Conversely, assume that $A = (A, +, \leq, *)$ is a commutative ℓ-group in which $a * b = |a-b|$. Then $a \geq 0$ implies that $a*0 = |a| = a \cup (-a) = a \cup 0 \cup (-a) = a$ (since $-a \leq 0$). Also $a*0 = |a| = a \cup (-a) \geq a$ for every a in A. Clearly for each a in A, the mapping $x \mapsto a + x$ is onto and $(a+x) * (a+y) = |(a+x) - (a+y)| = |x-y| = x * y$ for all x, y in A. Hence for each $a \in A$, the mapping $x \mapsto a + x$ is an isometry from A onto A. By Theorem 3 ([1], page 293) it follows that $2x \geq 0$ implies $x \geq 0$ for all x in A. Also for each $a \in A$, $a + a*0 = a + |a| = (a \cup 0 + a \cap 0) + (a \cup 0 - a \cap 0) = 2(a \cup 0)$. Thus A satisfies all the conditions

(1.3.1) through (1.3.5). This completes the proof.

Remark 1.4. The conditions (1.3.1) and (1.3.3) are independent of the remaining conditions (1.3.2), (1.3.4) and (1.3.5) on an autometrized algebra, in view of the following examples.

Example 1.4.1. Let $G = G_1 \circ G_2$ be the lexicographic product of non-trivial abelian ℓ-groups G_1 and G_2 where G_1 is not a chain. Define $(a_1, a_2) * (b_1, b_2) = (|a_1-b_1|, |a_2-b_2|)$ for all (a_1, a_2) and (b_1, b_2) in G. Then it can be easily seen that $G = (G, +, \leq, *)$, where $+$ is the componentwise addition and $\leq$ is the lexicographic ordering, is an autometrized algebra in which all the conditions (1.3.2) through (1.3.5) are satisfied while (1.3.1) is not satisfied. For, let $a_1 > 0$ in G_1 and $a_1 > 0$ in G_2. Then $(a_1, -a_2) > (0, 0)$. But $(a_1, -a_2) * (0, 0) = (|a_1|, |-a_2|) = (|a_1|, |a_2|) = (a_1, a_2) \neq (a_1, -a_2)$ (since $a_2 \neq -a_2$).

Example 1.4.2. Let A be the set of all nonnegative real numbers with usual addition $+$ and usual ordering $\leq$. Define $a*b = |a-b|$ for all a, b in A. Then it can be easily verified that $A = (A, +, \leq, *)$ is an autometrized algebra which satisfies all the conditions (1.3.1) through (1.3.5) except (1.3.2) because, for any a in A, the mapping $x \mapsto a + x$ is not onto.

We recall that a mapping $f: A \to A$ is called a *contraction mapping* with respect to a metric $*$ on A if and only if $f(x)*f(y) \leq x*y$ for all x and y in A.

The following theorem is a characterization of DRℓ-semigroups with

least element among the class of autometrized algebras.

Theorem 1.5. *Let* $A = (A, +, \leq, *)$ *be an autometrized algebra. Then* A *is a commutative DRℓ-semigroup with least element* 0 *in which* $a*b = (a-b) \cup (b-a)$ *if and only if* A *satisfies the following:*

(1.5.1) $(A, \leq)$ *is a lattice with* 0 *as the least element (where the lattice operations are denoted by* $\cup$ *and* $\cap$*) such that* $a + (b \cup c) = (a+b) \cup (a+c)$ *and* $a + (b \cap c) = (a+b) \cap (a+c)$ *for all* a, b, c *in* A.

(1.5.2) *For each* $a \in A$, *the mappings* $x \mapsto a + x$, $x \mapsto a \cup x$ *and* $x \mapsto a \cap x$ *are contractions.*

(1.5.3) $a*(a \cap b) + (a \cap b) = a$ *for all* a, b *in* A.

Proof. Assume that $A = (A, +, \leq, *)$ satisfies (1.5.1) through (1.5.3). We observe that $a*0 = a$ for all a in A since $a = a*(a \cap 0) + (a \cap 0)$ (by 1.5.3) $= a*0 + 0 = a*0$. From (1.5.1) it follows that $A = (A, +, \leq)$ is a commutative lattice ordered semigroup with identity 0 which is also the least element of A. Define $a-b = (a \cup b)*b$ for all a and b in A. Let $a, b \in A$. $(a-b) + b = (a \cup b)*b + b*0 \geq (a \cup b)*0 = a \cup b \geq a$. Let $x + b \geq a$. Then $a-b = (a \cup b)*b = (a \cup b)*((a \cap b) \cup b) \leq a*(a \cap b)$ (by 1.5.2) $= ((x+b) \cap a)*(a \cap b) \leq (x+b)*b$ (by 1.5.2) $= (x+b)*(0+b) \leq x*0$ (by 1.5.2) $= x$. Thus $a-b$ is the least of all x such that $x + b \leq a$. Also for any a, b in A we have $(a-b) \cup 0 + b = ((a \cup b)*b) \cup 0 + b = (a \cup b)*b + b = (a \cup b)*((a \cup b) \cap b) + (a \cup b) \cap b = a \cup b$ (by 1.5.3). Further, $a-a = (a \cup a)*a = a*a = 0$ for every a in A. Hence $A = (A, +, \leq, -)$ is a commutative DRℓ-semigroup with least element 0. Write $a \Theta b = (a-b) \cup (b-a)$. Now $a-b = (a \cup b)*b \leq a*b$ (by 1.5.2) and $b-a = (a \cup b)*a$

$\leq b*a = a*b$ (by 1.5.2). This implies $a\Theta b \leq a*b$. Also $a*b \leq a*(a \cup b) + (a \cup b)*b = (b-a) + (a-b) = (b-a) \cup 0 + (a-b) \cup 0 = (b-a) \cup (a-a) + (a-b) \cup (a-a) = (b \cup a-a) + (a - b \cap a) = b \cup a - b \cap a = (a-b) \cup (b-a) = a\Theta b$. Thus $a*b = a\Theta b$ for all a, b in A. Conversely assume that A is a DRℓ-semigroup with least element 0 in which $a*b = (a-b) \cup (b-a)$. Then (1.5.1) follows from the definition of a DRℓ-semigroup, (1.5.3) follows from Lemma 14 of [6] by putting $a \cap b$ in place of b. Now let $a \in A$. $(a+x)*(a+y) = ((a+x) - (a+y)) \cup ((a+y) - (a+x)) \leq (x-y) \cup (y-x) = x*y$, $(a \cup x)*(a \cup y) = (a \cup x - a \cup y) \cup (a \cup y - a \cup x) \leq (x-y) \cup (y-x) = x*y$ and $(a \cap x)*(a \cap y) = (a \cap x - a \cap y) \cup (a \cap y - a \cap x) \leq (x-y) \cup (y-x)$ for all x and y in A. Thus A satisfies (1.5.2) also. This completes the proof.

§ 2. We recall now the definition of a lattice ordered autometrized algebra [7].

Definition 2.1. A system $A = (A, +, \leq, *)$ is said to be a *lattice ordered autometrized algebra* if and only if

(2.1.1) $(A, +, \leq)$ is a commutative lattice ordered semigroup with identity 0 i.e., $(A, +)$ is a commutative semigroup with identity 0 and $(A, \leq)$ is a lattice (where the lattice operations are denoted by $\cup$, $\cap$) such that

$$a + (b \cup c) = (a+b) \cup (a+c)$$

$$a + (b \cap c) = (a+b) \cap (a+c)$$

for all a, b, c in A

(2.1.2) $*$ is a metric operation on A, i.e., $*$ is a mapping from $A \times A$ to A satisfying M1, M2 and M3 of Definition 1.1.

We now introduce

Definition 2.2. A lattice ordered autometrized algebra $A = (A, +, \leq, *)$ is called a *representable autometrized algebra* if and only if A satisfies the following:

(2.2.1) $(A, +, \leq, *)$ is a semiregular autometrized algebra.

(2.2.2) For every $a \in A$, all the mappings $x \mapsto a+x$, $x \mapsto a \cup x$, $x \mapsto a \cap x$ and $x \mapsto a*x$ are contractions.

Remark 2.3. Commutative DRℓ-semigroup (consequently Boolean algebras and commutative ℓ-groups) are representable autometrized algebras, but the following example shows that the class of representable autometrized algebras is wider than the class of DRℓ-semigroups.

Example 2.3.1. Let A be the set consisting of elements $0, a, b, c, d, e$. Define $0 < a < b < c < d < e$. Define $+$ and $*$ by the following tables

$+$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	b	b	e	e	e
b	b	b	b	e	e	e
c	c	e	e	e	e	e
d	d	e	e	e	e	e
e	e	e	e	e	e	e

$*$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	0	a	c	c	c
b	b	a	0	c	c	c
c	c	c	c	0	a	a
d	d	c	c	a	0	a
e	e	c	c	a	a	0

It is routine to verify that $A = (A, +, \leq, *)$ is a representable autometrized algebra. If there is a binary operation $-$ such that $A = (A, +, \leq, -)$ is a DRℓ-semigroup, then that operation $-$ must be as follows.

-	0	a	b	c	d	e
0	0	0	0	0	0	0
a	a	0	0	0	0	0
b	b	a	0	0	0	0
c	c	c	c	0	0	0
d	d	c	c	a	0	0
e	e	c	c	a	a	0

Note that $(d\text{-}a) \cup 0 + a = (c \cup 0) + a = c + a = e$ whereas $d \cup e = d \neq e$. Hence the condition "$(x\text{-}y) \cup 0 + y = x \cup y$ for any x, y" of a DRℓ-semigroup fails. Thus A is not a DRℓ-semigroup.

Remark 2.4. Any representable autometrized algebra is an abstract algebra of species (2, 2, 2, 2, 0). This class of autometrized algebras is closed under the formation of subalgebras and direct products. In fact, if $\{A_i\}_{i\in I}$ is any family of representable autometrized algebras and if $A = \prod_{i\in I} A_i$, then for any a, b in A, if we define the operations $+$, $\cup$, $\cap$ and $*$ componentwise (i.e., $(a+b)_i = a_i + b_i$, $(a \cup b)_i = a_i \cup b_i$, $(a \cap b)_i = a_i \cap b_i$ and $(a*b)_i = a_i * b_i$ for any i in I), then A is a representable autometrized algebra. But the class of representable autometrized algebras is not an "equational class" since it is not closed under the formation of homomorphic images. Specifically we give below an example of a representable autometrized algebra A and an algebra B of species (2, 2, 2, 2, 0) and a mapping $f\colon A \to B$ preserving all the operations but yet $f(A)$ is not an autometrized algebra.

Example 2.4.1. Let $A = Z \cup \{u, v\}$ where Z is the set of all inte-

gers and u and v are elements which are not in Z. Define $+$ in A as follows: $a + b$ = usual sum of a and b for all a, b in Z, $a + u = u = u + a$ for all a in Z, $a + v = v = v + a$ for all a in Z, $u + v = u = v + u$, $u + u = u$ and $v + v = v$. Define $\leq$ in A as follows: For the elements in Z, let $\leq$ be the usual ordering. Define $u < a < v$ for all a in Z. Define $*$ in A as follows: $a*b = |a-b|$ for all a, b in Z, $a*u = v = u*a$ for all a in Z, $a*v = v = v*a$ for all a in Z, $u*v = v = v*u$ and $u*u = 0 = v*v$. Then it can be verified that $A = (A, +, \leq, *)$ is a presentable autometrized algebra. Let $B = \{u, 0\}$. Define $u < 0$. Define $+$ and $*$ in B by the following tables

$+$	0	u
0	0	u
u	u	u

$*$	0	u
0	0	0
u	0	0

Then it can be seen that $B = (B, +, \cup, \cap, *)$ is also an algebra of the same species $(2, 2, 2, 2, 0)$ as A. Define $f: A \to B$ by $f(u) = u$, $f(v) = 0$ and $f(a) = 0$ for all a in Z. Then it can be verified that f is a homomorphism from A onto B (i.e., a mapping preserving all the operations $+$, $\cup$, $\cap$, $*$ and 0). But B is not even an autometrized algebra since $u*0 = 0$ whereas $u \neq 0$.

Remark 2.4.2. Every representable autometrized algebra is a seminormal autometrized algebra [3]. We recall the following definitions of an ideal and a congruence relation in an autometrized algebra from [4].

Definition 2.5. A nonempty subset I of an autometrized algebra

$A = (A, +, \leq, *)$ is called an *ideal* if and only if

(I1) $a \in I$, $b \in I$ imply $a+b \in I$

(I2) $a \in I$, $b \in A$, $b*0 \leq a*0$ imply $b \in I$.

Definition 2.6. An equivalence relation Θ on an autometrized algebra $A = (A, +, \leq, *)$ is called a *congruence relation* if and only if

(C1) $a \equiv b(\Theta)$, $c \equiv d(\Theta)$ imply that $a+c \equiv b+d(\Theta)$ for any a, b, c, d in A

(C2) $a \equiv b(\Theta)$, $c \equiv d(\Theta)$ imply that $a*c \equiv b*d(\Theta)$ for any a, b, c, d in A

(C3) $a \equiv b(\Theta)$ and $x*y \leq a*b$ imply that $x \equiv y(\Theta)$ for any a, b, x, y in A.

Remark 2.6.1. If Θ is a congruence relation on a representable autometrized algebra $A = (A, +, \leq, *)$ then one can very easily observe that (C2) follows from (C1) and (C3). Further, in this case one can deduce the following:

(C4) $a \equiv b(\Theta)$, $c \equiv d(\Theta)$ imply that $a \cup c \equiv b \cup d(\Theta)$ for any a, b, c, d in A

(C5) $a \equiv b(\Theta)$, $c \equiv d(\Theta)$ imply that $a \cap c \equiv b \cap d(\Theta)$ for any a, b, c, d in A.

Now we have the following

Theorem 2.7. *The lattice of ideals of a representable autometrized algebra* $A = (A, +, \leq, *)$ *is isomorphic with the lattice of congruence relations of* A.

Proof. Obvious from Theorem 3 in [3] and Remark (2.4.2) above.

If Θ is any congruence relation on a set S, as usual S/Θ denotes the quotient set (i.e., the set of all equivalence classes) and for any x in S, $\overline{x}$ denotes the equivalence class containing x.

Theorem 2.8. *Let Θ be a congruence relation on a representable autometrized algebra. For any $\overline{a}$, $\overline{b}$ in A/Θ define $\overline{a} + \overline{b} = \overline{a + b}$, $\overline{a} \cup \overline{b} = \overline{a \cup b}$, $\overline{a} \cap \overline{b} = \overline{a \cap b}$ and $\overline{a} * \overline{b} = \overline{a*b}$. Also define $\overline{a} \leq \overline{b}$ if and only if $\overline{a} \cup \overline{b} = \overline{b}$. Then $(A/\Theta, +, \leq, *)$ is a representable autometrized algebra.*

Proof. Let Θ be a congruence relation on a representable autometrized algebra $A = (A, +, \leq, *)$. To show that the operations $+$, $\cup$, $\cap$ and $*$ are well defined, let $\overline{a} = \overline{c}$ and $\overline{b} = \overline{d}$. Then $a \equiv c(\Theta)$ and $b \equiv d(\Theta)$. From (C1), (C4), (C5) and (C2) it follows respectively that $\overline{a+b} = \overline{c+d}$, $\overline{a \cup b} = \overline{c \cup d}$, $\overline{a \cap b} = \overline{c \cap d}$ and $\overline{a*b} = \overline{c*d}$. Hence $+$, $\cup$, $\cap$ and $*$ are well defined. As a consequence, it follows that $\leq$ is also well defined. Since $\overline{a} + \overline{b} = \overline{a+b} = \overline{b + a} = \overline{b} + \overline{a}$, $\overline{a} + \overline{0} = \overline{a+0} = \overline{a}$ and $\overline{a} + (\overline{b} + \overline{c})$ $= \overline{a} + \overline{b+c} = \overline{a + (b + c)} = \overline{(a+b)+c} = \overline{a+b} + \overline{c} = (\overline{a} + \overline{b}) + \overline{c}$ for any $\overline{a}$, $\overline{b}$ $\overline{c}$ in A/Θ, it follows that $(A/\Theta, +)$ is a commutative semigroup with additive identity $\overline{0}$. Since $\overline{a} \cup \overline{a} = \overline{a}$, $\overline{a} \cup \overline{b} = \overline{a \cup b} = \overline{b \cup a} = \overline{b} \cup \overline{a}$, $\overline{a} \cup (\overline{b} \cup \overline{c})$ $= \overline{a} \cup \overline{b \cup c} = \overline{a \cup (b \cup c)} = \overline{(a \cup b) \cup c} = \overline{a \cup b} \cup \overline{c} = (\overline{a} \cup \overline{b}) \cup \overline{c}$, $\overline{a} \cap \overline{b} = \overline{a \cap b}$ $= \overline{b \cap a} = \overline{b} \cap \overline{a}$, $\overline{a} \cap (\overline{b} \cap \overline{c}) = \overline{a} \cap \overline{b \cap c} = \overline{a \cap (b \cap c)} = \overline{(a \cap b) \cap c} = \overline{a \cap b} \cap \overline{c}$ $= (\overline{a} \cap \overline{b}) \cap \overline{c}$, $\overline{a} \cup (\overline{a} \cap \overline{b}) = \overline{a} \cup \overline{a \cap b} = \overline{a \cup (a \cap b)} = \overline{a}$ and $\overline{a} \cap (\overline{a} \cup \overline{b}) = \overline{a} \cap \overline{a \cup b}$ $= \overline{a \cap (a \cup b)} = \overline{a}$ for all $\overline{a}$, $\overline{b}$ and $\overline{c}$ in A/Θ, it follows that $(A/\Theta, \cup, \cap)$ is a lattice. Also $\overline{a} + \overline{b} \cup \overline{c} = \overline{a} + \overline{b \cup c} = \overline{a + b \cup c} = \overline{(a+b) \cup (a+c)}$ $= \overline{a+b} \cup \overline{a+c} = (\overline{a} + \overline{b}) \cup (\overline{a} + \overline{c})$ and $\overline{a} + \overline{b} \cap \overline{c} = \overline{a} + \overline{b \cap c} = \overline{a + b \cap c}$ $= \overline{(a + b) \cap (a + c)} = \overline{a + b} \cap \overline{a + c} = (\overline{a} + \overline{b}) \cap (\overline{a} + \overline{c})$ for all $\overline{a}$, $\overline{b}$ and

$\overline{c}$ in A/Θ. Hence $(A/\Theta, +, \leq)$ is a lattice ordered semigroup with identity $\overline{0}$. To show that $*$ is a metric on A/Θ, we observe that $\overline{a} * \overline{b} \geq \overline{0}$ for all $\overline{a}$, $\overline{b}$ in A/Θ since we have $0 \cup (a*b) = a*b$ which implies that $\overline{0} \cup \overline{a*b} = \overline{a*b}$ which implies $\overline{0} \leq \overline{a} * \overline{b}$. If $\overline{a} * \overline{b} = \overline{0}$, then since $\overline{a * b} = \overline{0}$, we have $a * b \equiv 0\ (\Theta)$ which yields $a \equiv b\ (\Theta)$ (since $a * b = (a * b) * 0$ implies $a \equiv b\ (\Theta)$) or $\overline{a} = \overline{b}$. Let $\overline{a}$, $\overline{b}$, $\overline{c}$ be in A/Θ. Then from $a * b \leq a * c + c * b$, we have $(a * b) \cup (a * c + c * b) = a * c + c * b$, which implies

$$(\overline{a} * \overline{b}) \cup (\overline{a} * \overline{c} + \overline{c} * \overline{b}) = \overline{a * b} \cup \overline{a * c + c * b}$$
$$= \overline{a * c + c * b}$$
$$= \overline{a} * \overline{c} + \overline{c} * \overline{b}$$

or equivalently $\overline{a} * \overline{b} \leq \overline{a} * \overline{c} + \overline{c} * \overline{b}$. Further if $\overline{a} \geq \overline{0}$, then since $\overline{a \cup 0} = \overline{a} \cup \overline{0} = \overline{a}$, we have $a \cup 0 \equiv a\ (\Theta)$, and hence by (C2),

$$a * 0 \equiv (a \cup 0) * 0\ (\Theta) \equiv a \cup 0\ (\Theta)$$
$$\equiv a\ (\Theta) \quad \text{(since } A \text{ is semiregular)},$$

which implies $\overline{a} * \overline{0} = \overline{a * 0} = \overline{a}$. Hence $(A/\Theta, +, \leq, *)$ is a semiregular lattice ordered autometrized algebra. For any $\overline{a}$, $\overline{x}$ and $\overline{y}$ in A/Θ, we have $(\overline{a} + \overline{x})*(\overline{a} + \overline{y}) \leq \overline{x} * \overline{y}$ since $(a+x)*(a+y) \leq x*y$, $(\overline{a} \cup \overline{x})*(\overline{a} \cup \overline{y}) \leq \overline{x} * \overline{y}$ since $(a \cup x)*(a \cup y) \leq x*y$, $(\overline{a} \cap \overline{x})*(\overline{a} \cap \overline{y}) \leq \overline{x} * \overline{y}$ since $(a \cap x)*(a \cap y) \leq x*y$ and $(\overline{a} * \overline{x})*(\overline{a} * \overline{y}) \leq \overline{x} * \overline{y}$ since $(a*x)*(a*y) \leq x*y$. Thus for any $\overline{a}$ in A/Θ, all the mappings $\overline{x} \mapsto \overline{a} + \overline{x}$, $\overline{x} \mapsto \overline{a} \cup \overline{x}$, $\overline{x} \mapsto \overline{a} \cap \overline{x}$ and $\overline{x} \mapsto \overline{a} * \overline{x}$ are contractions. Hence $(A/\Theta, +, \leq, *)$ is a representable autometrized algebra.

Following the well known concepts in Universal algebra [2], we introduce the following:

Definition 2.9. Let $\{A_i \mid i \in I\}$ be a family of representable autometrized algebras. Let $S \subset \Pi(A_i \mid i \in I)$ be such that S is a subalgebra of the direct product. S is called a *subdirect product* of A_i, $i \in I$, if $Se_i = A_i$ for all $i \in I$, where e_i is the i-th projection.
Also,

Definition 2.10. A representable autometrized algebra A is called *subdirectly irreducible* if $\bigwedge_{i \in I} \Theta_i = \omega$ ($\Theta_i \in C(A)$, the complete lattice of congruence relation on A) implies that $\Theta_i = \omega$ for some $i \in I$. (Here ω denotes the diagonal of A viz., $\{(a, a) \mid a \in A\}$).

Hereafter, $A = (A, +, \leq, *)$ stands for a representable autometrized algebra.

Now, we prove the following

Theorem 2.11. *A is a subdirect product of subdirectly irreducible representable autometrized algebras.*

We need the following lemmas.

Lemma 2.11.1. *Let Θ be a congruence relation on A. Then there exists a one to one correspondence between congruence relations of A containing Θ and congruence relations of A/Θ.*

Proof. Let ϕ be a congruence relation on A containing Θ. Define ϕ/Θ on A/Θ by $[a]^\Theta \equiv [b]^\Theta$ (ϕ/Θ) if and only if $a \equiv b$ (Θ) (Here $[x]^\Theta$ denotes the equivalence class containing x relative to Θ). It is easy to verify that ϕ/Θ is an equivalence relation on A/Θ. Now let

$[a]^\Theta \equiv [b]^\Theta$ (ϕ/Θ) and $[c]^\Theta \equiv [d]^\Theta$ (ϕ/Θ). Then $a \equiv b$ (ϕ) and $c \equiv d$ (ϕ). Hence $a + c \equiv b + d$ (ϕ). This implies $[a]^\Theta + [c]^\Theta = [a + c]^\Theta \equiv [b + d]^\Theta$ $(\phi/\Theta) = [b]^\Theta + [d]^\Theta$. Let $[a]^\Theta \equiv [b]^\Theta$ (ϕ/Θ) and $[x]^\Theta * [y]^\Theta \leq [a]^\Theta * [b]^\Theta$. Then $a \equiv b$ (ϕ) and $[x*y]^\Theta \leq [a*b]^\Theta$. This implies that $a*b \equiv 0$ (ϕ) (by (C2)) and $(x*y) \cup (a*b) \equiv (a*b)$ (Θ) and this implies that $[(x*y) \cup (a*b)] * (a*b) \equiv 0$ (Θ). Since $\Theta \leq \phi$, it follows that $a*b \equiv 0$ (ϕ) and $[(x*y) \cup (a*b)] * (a*b) \equiv 0$ (ϕ) so that $a*b + [(x*y) \cup (a*b)] * (a*b) + a*b \equiv 0$ (ϕ). Hence

$$
\begin{aligned}
x*y &= (x*y)*0 \quad \text{(by semiregularity of } * \text{ in } A\text{)} \\
&\leq (x*y)*[(x*y) \cup (a*b)] + [(x*y) \cup (a*b)] * 0 \\
&\leq [(x*y) \cup 0] * [(x*y) \cup (a*b)] + [(x*y) \cup (a*b)] * (a*b) + (a*b) * 0 \\
&\leq 0*(a*b) + [(x*y) \cup (a*b)] * (a*b) + (a*b)*0 \quad \text{(by (2.1.2))} \\
&= [(a*b) + ((x*y) \cup (a*b)) * (a*b) + (a*b)] * 0
\end{aligned}
$$

(by semiregularity of $*$) which implies $x \equiv y$ (ϕ) so that $[x]^\Theta \equiv [y]^\Theta$ (ϕ/Θ). Thus ϕ/Θ is a congruence relation on A/Θ.

Now let $\overline{\phi}$ be a congruence relation on A/Θ. Define $a \equiv b$ (ϕ_Θ) if and only if $[a]^\Theta \equiv [b]^\Theta$ $(\overline{\phi})$. Clearly ϕ_Θ is an equivalence relation having the substitution property for $+$. Let $a \equiv b$ (ϕ_Θ) and $x*y \leq a*b$. Then $[a]^\Theta \equiv [b]^\Theta$ $(\overline{\phi})$ and $(x*y) \cup (a*b) = a*b$ which implies $[x]^\Theta * [y]^\Theta \leq [a]^\Theta * [b]^\Theta$. Hence $[x]^\Theta \equiv [y]^\Theta$ $(\overline{\phi})$. Hence $x \equiv y$ (ϕ_Θ). Thus ϕ_Θ is a congruence relation on A. Also $\Theta \leq \phi_\Theta$. For if $x \equiv y$ (Θ) then $[x]^\Theta = [y]^\Theta$, and hence $[x]^\Theta \equiv [y]^\Theta$ $(\overline{\phi})$ or $x \equiv y$ (ϕ_Θ). Let ϕ be a congruence relation of A containing Θ. Then since each of $x \equiv y$ (ϕ) and $x \equiv y$ (ϕ_Θ) is equivalent to $[x]^\Theta \equiv [y]^\Theta$ (ϕ/Θ), we see that $x \equiv y$ (ϕ) and $x \equiv y$ (ϕ_Θ) are equivalent. It follows that $\phi = \phi_\Theta$. Moreover, if $\overline{\phi}$ is a congruence relation of A/Θ, then $[x]^\Theta \equiv [y]^\Theta$ $(\overline{\phi})$ and $[x]^\Theta \equiv [y]^\Theta$

(ϕ/Θ) are equivalent, since each of them is equivalent to $x \equiv y\ (\phi_\Theta)$; and so we have $\overline{\phi} = \phi/\Theta$. This completes the proof.

Lemma 2.11.2. *Let* $\{\Theta_i\}_{i\in I}$ *be a family of congruence relations on* A *such that* $\bigwedge_{i\in I} \Theta_i = \omega$ *(the diagonal of* A*). Then* A *is isomorphic to a subdirect product of representable autometrized algebras* A/Θ_i, $i \in I$.

Proof. Define $f_a \in \Pi(A/\Theta_i \mid i \in I)$ by $if_a = [a]^{\Theta i}$ for $i \in I$ (for each $x \in A$, $[x]^{\Theta i}$ stands for the equivalence class containing x relative to the equivalence relation Θ_i). Let $A' = \{f_a \mid a \in A\}$. Since if_{a+b} $[a + b]^{\Theta i} = [a]^{\Theta i} + [b]^{\Theta i} = if_a + if_b = i(f_a + f_b)$ for any $i \in I$, we have $f_{a+b} = f_a + f_b$. Similarly we can show that $f_{a\cup b} = f_a \cup f_b$, $f_{a\cap b} = f_a \cap f_b$ and $f_{a*b} = f_a * f_b$. Hnece A' is a subalgebra of the direct product. Since $f_a e_i = if_a = [a]^{\Theta i}$, we have $A'e_i = A/\Theta_i$ (e_i denotes the i-th projection). Hence A' is a subdirect product of A/Θ_i. Define ψ: $A \to A'$ by $a\psi = f_a$. Clearly ψ is an epimorphism. To show that ψ is one to one, let $a\psi = b\psi$. Then $f_a = f_b$. Hence $if_a = if_b$ for every $i \in I$. Thus $a \equiv b\ (\Theta_i)$ for every $i \in I$. Hence $a \equiv b\ (\bigwedge_{i\in I} \Theta_i)$. Since $\bigwedge_{i\in I} \Theta_i = \omega$, it follows that $a = b$. Hence ψ is one to one. This completes the proof.

Lemma 2.11.3. *Let* $\{\Theta_i\}_{i\in I}$ *be a directed family of congruence relations on* A. *Then* $\bigcup(\Theta_i \mid i \in I) = \vee(\Theta_i \mid i \in I)$.

Proof. Routine as for Lemma 3, page 51, [2].

Lemma 2.11.4. *If* $a, b \in A$ *and* $a \neq b$, *then there exists a congruence relation* $\Theta(a, b)$ *such that* $a \not\equiv b\ (\Theta(a, b))$ *and* $\Theta(a, b)$ *is maximal with respect to this property.*

Proof. Let $P = \{\phi \in C(A) \mid a \not\equiv b\ (\phi)\}$. P is nonempty since $\omega \in P$. Let C be a chain in P. Then $\Theta = \bigvee(\phi \mid \phi \in C)$ is a congruence relation on A. By Lemma 2.11.3, we have $\Theta = \bigcup(\phi \mid \phi \in C)$. Hence $x \equiv y\ (\Theta)$ if and only if $x \equiv y\ (\phi)$ for some $\phi \in C$. Hence $a \not\equiv b\ (\Theta)$ and so $\Theta \in P$. Now, by Zorn's lemma, it follows that P has a maximal element $\Theta(a, b)$.

Lemma 2.11.5. *A is subdirectly irreducible if and only if $C(A)$ has one and only one atom which is contained in every congruence relation other than ω.*

Proof. Routine (see [2], p. 124).

Now by using the same argument as in [2] (see p.]24), the proof of Theorem 2.11 is clear.

Now, we prove the following main theorem.

Theorem 2.12. *A lattice ordered autometrized algebra A is a subdirect product of representable totally ordered autometrized algebras if and only if A is representable and for all a, b in A,* $(a*(a \cup b)) \cap (b*(a \cup b)) = 0$.

We need the following lemmas.

Lemma 2.12.1. *If $a > 0$ in a representable autometrized algebra $A = (A, +, \leq, *)$, then the set*

$$J_a = \{x \in A \mid x*0 \leq na \text{ for some positive integer } n\}$$

is an ideal of A.

Proof. Since $0*0 = 0 < a$, we have $0 \in J_a$. Hence J_a is nonempty.

Now, if $x, y \in J_a$ then $x*0 \leq ma$ and $y*0 \leq na$ for some positive integers m and n, and so $(x+y)*0 \leq (x+y)*y + y*0 \leq x*0 + y*0$ (by 2.1.2) $\leq ma + na = (m+n)a$, which shows $x + y \in J_a$. Also if $x \in J_a$ and $y*0 \leq x*0$ then we have $y*0 \leq x*0 \leq na$ for some positive integer n proving $y \in J_a$. Hence J_a is an ideal of A.

Lemma 2.12.2. *Let* $A = (A, +, \leq, *)$ *be a subdirectly irreducible representable autometrized algebra such that* $(a*(a \cup b)) \cap (b*(a \cup b)) = 0$ *for all* a, b *in* A. *Then* A *is a chain.*

Proof. Suppose A is not a chain. Then there exists a pair of incomparable elements a, b in A so that $a \cup b \neq a$ and $a \cup b \neq b$. Hence $(a \cup b)*a \neq 0$ and $(a \cup b)*b \neq 0$. This implies $(a \cup b)*a > 0$ and $b*(a \cup b) > 0$. Let $J_1 = J_{a*(a \cup b)}$ and $J_2 = J_{b*(a \cup b)}$ (with the same notation as in Lemma 2.12.1). By Lemma 2.12.1, it follows that J_1 and J_2 are ideals of A. Since $(a*(a \cup b))*0 = a*(a \cup b)$ (by semiregularity of $*$), we see that $a*(a \cup b) \in J_1$ and since $a*(a \cup b) \neq 0$, it follows that $J_1 \neq \{0\}$. Similarly we can show that $J_2 \neq \{0\}$. Let $x \in J_1 \cap J_2$. Then $x \in J_1$ and $x \in J$. Hence $x*0 \leq m(a*(a \cup b))$ and $x*0 \leq n(b*(a \cup b))$ for some positive integers m and n. This implies $x*0 \leq (m+n)(a*(a \cup b))$ and $x*0 \leq (m+n)(b*(a \cup b))$. Hence $x*0 \leq ((m+n)(a*(a \cup b))) \cap ((m+n)(b*(a \cup b)))$ $= 0$ (since $(a*(a \cup b)) \cap (b*(a \cup b)) = 0$ and $(A, +, \leq)$ is a lattice ordered semigroup). This implies $x = 0$. Thus $J_1 \cap J_2 = \{0\}$ whereas $J_1 \neq \{0\}$ and $J_2 \neq \{0\}$. Since from Theorem 2.7 there exist congruence relations Θ_1 and Θ_2 respectively corresponding to J_1 and J_2, it follows that $\Theta_1 \wedge \Theta_2 = \omega$ whereas $\Theta_1 \neq \omega$ and $\Theta_2 \neq \omega$. Hence A is subdirectly reducible. This completes the proof.

Lemma 2.12.3. *Let* A *be a representable autometrized algebra such that* $(a*(a \cup b)) \cap (b*(a \cup b)) = 0$ *for all* a, b *in* A. *Let* Θ *be a congruence relation on* A. *Then* A/Θ *is a representable autometrized algebra and*

$$(\overline{a} * (\overline{a} \cup \overline{b})) \cap (\overline{b} * (\overline{a} \cup \overline{b})) = \overline{0} \quad \textit{for all} \quad \overline{a}, \overline{b} \quad \textit{in} \quad A/\Theta$$

($\overline{x}$ *stands for equivalence class containing* x *relative to* Θ).

Proof. By Theorem 2.8, it is clear that $(A/\Theta, +, \leq, *)$ is a representable autometrized algebra. For all $\overline{a}, \overline{b}$ in A/Θ, we have

$$\begin{aligned}(\overline{a} * (\overline{a} \cup \overline{b})) \cap (\overline{b} * (\overline{a} \cup \overline{b})) &= (\overline{a} * \overline{(a \cup b)}) \cap (\overline{b} * \overline{a \cup b}) \\ &= \overline{a * (a \cup b)} \cap \overline{b * (a \cup b)} \\ &= \overline{(a * (a \cup b)) \cap (b * (a \cup b))} \\ &= \overline{0} \quad \text{(by hypothesis)},\end{aligned}$$

proving the lemma.

Proof of Theorem 2.12. Suppose A is a representable autometrized algebra and $(a*(a \cup b)) \cap (b*(a \cup b)) = 0$ for all a, b in A. Then by Theorem 2.11 and Lemmas 2.12.2 and 2.12.3, it is clear that A is a subdirect product of representable totally ordered autometrized algebras. For the other part, assume that A is a subdirect product of representable totally ordered autometrized algebras A_i, $i \in I$. Since A is a subalgebra of the direct product of representable autometrized algebras A_i, $i \in I$, it follows that A is a representable autometrized algebra. Let $a, b \in A$. Let a_i and b_i denote the i-th components of a and b respectively in the subdirect product. Since each A_i is a chain, $a_i \cup b_i = a_i$ or b_i for every $i \in I$. This implies $a_i * (a_i \cup b_i) = 0_i$ or $b_i * (a_i \cup b_i) = 0_i$ for every $i \in I$. Thus $(a*(a \cup b)) \cap (b*(a \cup b)) = 0$ for all a, b in A.

This completes the proof.

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