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On Aliasing in Fractional 2^m Factorial Designs of Even Resolution

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1. Introduction. Consider an experiment with m factors each at two levels. An assembly or treatment combination is represented by $(j_1, j_2, \cdots, j_m)$ where j_k, the level of kth factor, equals 0 or 1. As unknown effects, θ_ϕ, θ_t, and in general $\theta_{t_1 \cdots t_k}$ denote the general mean, main effect of tth factor, and k-factor interaction of corresponding factors, respectively. For a fixed integer ℓ $(2 \leqq \ell \leqq m/2)$, let $\underline{\theta}$ be the $\nu \times 1$ vector composed of the effects up to ℓ-factor interactions and let $\underline{\theta}^*$ be the $\nu^* \times 1$ vector of the remaining effects, where $\nu = \sum_{\beta=0}^{\ell}\binom{m}{\beta}$ and $\nu^* = 2^m - \nu$, i.e.,

$$\underline{\theta}' = (\theta_\phi; \theta_1, \theta_2, \cdots, \theta_m; \theta_{12}, \cdots, \theta_{m-1m}; \cdots; \theta_{12\cdots\ell}, \cdots, \theta_{m-\ell+1\cdots m}),$$
$$\underline{\theta}^{*\prime} = (\theta_{12\cdots\ell+1}, \cdots, \theta_{m-\ell\cdots m}; \cdots; \theta_{12\cdots m}).$$

The expected value of the observation $y(j_1, \cdots, j_m)$ for an assembly $(j_1, \cdots, j_m)$ can then be expressed as

$$\mathrm{Exp}[y(j_1, \cdots, j_m)] = \underline{e}'\underline{\theta} + \underline{e}^{*\prime}\underline{\theta}^* , \tag{1.1}$$

where the elements of $(\underline{e}', \underline{e}^{*\prime})$ corresponding to $\theta_{t_1 t_2 \cdots t_k}$ are given by $d(j_{t_1})d(j_{t_2}) \cdots d(j_{t_k})$, and in particular the element to θ_ϕ is equal

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to 1 (see, e.g., Yamamoto, Shirakura and Kuwada [9]). Here $d(j)$ = -1 or 1 according as j = 0 or 1.

Let T be a fraction with N assemblies. (Note that T can be considered as a (0, 1) matrix of size $m \times N$ whose columns denote assemblies.) Consider the $N \times 1$ observation vector $\underline{y}_T$ of T whose elements are independent random variables with common variance σ^2. Then from (1.1), the expected value of $\underline{y}_T$ can be expressed as

$$\text{(1.2)} \qquad \operatorname{Exp}[\underline{y}_T] = E\,\underline{\theta} + E^*\,\underline{\theta}^*,$$

where E and E^* denote the $N \times \nu$ and $N \times \nu^*$ design matrices for $\underline{\theta}$ and $\underline{\theta}^*$ of T, respectively, whose elements are -1 or 1. Let $\underline{\theta}_0$ be the $p \times 1$ vector composed of the main effects, two-factor interactions, $\cdots$, and $(\ell-1)$-factor interactions, where $p = \nu - \binom{m}{\ell} - 1$. Then a fraction T is called a fractional 2^m factorial (briefly, 2^m-FF) design of resolution 2ℓ if $\underline{\theta}_0$ is estimable ignoring $\underline{\theta}^*$. It is known (cf. [5]) that T is a 2^m-FF design of resolution 2ℓ if and only if there exists a $p \times \nu$ matrix X satisfying

$$\text{(1.3)} \qquad XM = [\,\underline{0} \;:\; I_p \;;\; O\,] \quad (= C, \text{ say}),$$

where $M = E'E$ is called the information matrix of T. Here I_p denotes the identity matrix of order p, and $\underline{0}$ and O denote respectively the vector and matrix of appropriate sizes with all elements 0. In this case, the best linear unbiased estimate $\hat{\underline{\theta}}_0$ of $\underline{\theta}_0$ and its covariance matrix $\operatorname{Var}[\hat{\underline{\theta}}_0]$ are given by

$$\text{(1.4)} \qquad \hat{\underline{\theta}}_0 = XE'\underline{y}_T \quad \text{and} \quad \operatorname{Var}[\hat{\underline{\theta}}_0] = X_1\sigma^2,$$

respectively, where $X_1 = CX'$. However, under model (1.2), the expected value of $\hat{\underline{\theta}}_0$ becomes

$$\text{(1.5)} \qquad \mathrm{Exp}[\hat{\underline{\theta}}_0] = \underline{\theta}_0 + A\underline{\theta}^* ,$$

where $A = XE'E^*$. The matrix A gives an aliasing relation of $\underline{\theta}_0$ and $\underline{\theta}^*$ for $\hat{\underline{\theta}}_0$ in a 2^m-FF design of resolution 2ℓ. Therefore, A is called the alias matrix of T. In view of (1.5), Hedayat, Raktoe and Federer [1] have introduced the norm $\| A \| = \{\mathrm{tr}(AA')\}^{\frac{1}{2}}$ of A as a measure for determining alias goodness. Also, they have introduced the concept of alias balanced (AB) designs in order to classify designs.

In this paper, we investigate some properties of the norm $\| A \|$ for a 2^m-FF design of resolution 2ℓ. Furthermore, we introduce the concept of alias partially balanced (APB) designs as a generalization of AB designs. In Section 2, a connection between $\| A \|$ and $\mathrm{Var}[\hat{\underline{\theta}}_0]$ is given. Section 3 shows that orthogonal arrays of strength $2\ell-1$ yield AB designs and some balanced arrays of strength 2ℓ yield APB designs. Section 4 shows that $(2^m - 1)$ AB (or APB) designs can be generated from an AB (or APB) design by level permutations. To avoid repetition, therefore, assume throughout this paper that a fraction T is a 2^m-FF design of resolution 2ℓ (i.e., assume the existence of a matrix X in (1.3)). It may be noted that similar results for a 2^m-FF design of odd resolution have been given in [3, 7].

2. The norm $\| A \|$ and covariance matrix $\mathrm{Var}[\hat{\underline{\theta}}_0]$. Suppose that T is a fraction composed of n distinct assemblies $\underline{j}_q' = (j_{1q}, j_{2q}, \cdots, j_{mq})$, $(q = 1, \cdots, n)$, with each multiplicity r_q (i.e., $N = \sum_{q=1}^{n} r_q$). For any integer α with $1 \leqq \alpha \leqq s = \max\{r_1, \cdots, r_n\}$, let $T_\alpha = \{\underline{j}_q' ; r_q = \alpha,\ q = 1, \cdots, n\}$ and $N_\alpha = | T_\alpha |$, where $| \cdot |$ stands for the cardinality of a set. Consider

T_α as an $m \times N_\alpha$ matrix whose columns are assemblies. Then T can be expressed as

$$T = [T_1 : \underbrace{T_2 : T_2}_{2} : \cdots : \underbrace{T_s : \cdots : T_s}_{s}].$$

Note that if $T_\alpha = \phi$ as a set, then T_α vanishes from T. Suppose E_α is the $N_\alpha \times \nu$ design matrix for $\underline{\theta}$ of T_α. Then we have

Lemma 2.1. *For a fraction* T,

$$AA' = 2^m\{X_1 + XHX'\} - I_p, \tag{2.1}$$

where X_1 *is given in* (1.4) *and*

$$H = \textstyle\sum_{\alpha=2}^{s} \alpha(\alpha-1)E'_\alpha E_\alpha \tag{2.2}$$

Proof. Let E^*_α is the $N_\alpha \times \nu^*$ design matrix for $\underline{\theta}^*$ of T_α. Then we have

$$[E_\alpha : E^*_\alpha][E_\alpha : E^*_\alpha]' = 2^m I_{N_\alpha}, \quad [E_\alpha : E^*_\alpha][E_\beta : E^*_\beta]' = O, \ (\alpha \neq \beta).$$

Therefore, from (1.2) we have

$$\begin{aligned}[E : E^*][E : E^*]' &= EE' + E^*E^{*\prime} \\ &= 2^m \mathrm{diag}(I_{N_1}, G_2 \otimes I_{N_2}, \cdots, G_s \otimes I_{N_s}) \ (= D, \text{ say}),\end{aligned}$$

where G_r denotes the $r \times r$ matrix with all elements 1 and $\otimes$ denotes the Kronecker product. Thus it follows from (1.3) that

$$\begin{aligned}AA' &= XE'E^*E^{*\prime}EX' = XE'(D - EE')EX' \\ &= XE'DEX' - I_p .\end{aligned}$$

Since $E'DE = 2^m \sum_{\alpha=1}^{s} \alpha^2 E'_\alpha E_\alpha$ and $M = \sum_{\alpha=1}^{s} \alpha E'_\alpha E_\alpha$, it is easy to see that (2.1) holds.

Theorem 2.2. *For a fraction* T,

$$\| A \|^2 = 2^m\{\mathrm{tr}(\mathrm{Var}[\hat{\underline{\theta}}_0])/\sigma^2 + \mathrm{tr}(XHX')\} - p .$$

Proof. The proof follows immediately from Lemma 2.1.

Corollary 2.3. *For a fraction* T,

$$\| A \|^2 \geqq 2^m \ \mathrm{tr}(\mathrm{Var}[\hat{\underline{\theta}}_0])/\sigma^2 - p,$$

with equality if $s = 1$ (*i.e.*, $r_1 = r_2 = \cdots = r_n = 1$).

Proof. Since $\mathrm{tr}(XE_\alpha' E_\alpha X') \geqq 0$, the proof follows from Theorem 2.2.

Corollary 2.4. *Consider a fraction* T *such that* $r_1 = r_2 = \cdots = r_n = 1$. *If* T *minimizes the trace of covariance matrix of* $\hat{\underline{\theta}}_0$, *then it also minimizes the norm of alias matrix.*

Proof. For convenience, we write A_T and $\mathrm{Var}_T[\hat{\underline{\theta}}_0]$ for the alias matrix and covariance matrix of T, respectively. Suppose T_1 is another fraction with N assemblies. Then from the assumption and Corollary 2.3, it can easily be shown that

$$\begin{aligned} \| A_{T_1} \|^2 &\geqq 2^m \mathrm{tr}(\mathrm{Var}_{T_1}[\hat{\underline{\theta}}_0])/\sigma^2 - p \\ &\geqq 2^m \mathrm{tr}(\mathrm{Var}_T[\hat{\underline{\theta}}_0])/\sigma^2 - p \\ &= \|A_T\|^2 . \end{aligned}$$

This completes the proof.

3. AB and APB designs. Let $a(t_1 \cdots t_u;\ t_1' \cdots t_v')$, $(1 \leqq u < \ell,\ \ell+1 \leqq v \leqq m)$, be the element of A corresponding to effects $\theta_{t_1 \cdots t_u}$ and $\theta_{t_1' \cdots t_v'}$. Then (1.5) is equivalent to that for each $\hat{\theta}_{t_1 \cdots t_u}$ in $\hat{\underline{\theta}}_0$,

$$\mathrm{Exp}[\hat{\theta}_{t_1\cdots t_u}] = \theta_{t_1\cdots t_u} + \sum_{v=\ell+1}^{m}\sum_{\{t_1',\cdots,t_v'\}\in M_v} a(t_1\cdots t_u;t_1'\cdots t_v')\theta_{t_1'\cdots t_v'}$$

where M_k denotes the family of all subsets of $\{1,2,\cdots,m\}$ with cardinality k. We first define an AB design, due to [1].

Definition 1. *A fraction T is called an AB design if*

$$\{\sum_{v=\ell+1}^{m}\sum_{\{t_1',\cdots,t_v'\}\in M_v} a^2(t_1\cdots t_u;t_1'\cdots t_v')\}^{\frac{1}{2}}$$

is constant for all substes $\{t_1,\cdots,t_u\}$, $(1 \leqq u \leqq \ell-1)$.

As a generalization of Definition 1, we may make

Definition 2. *A fraction T is called an APB design if*

$$\{\sum_{v=\ell+1}^{m}\sum_{\{t_1',\cdots,t_v'\}\in M_v} a^2(t_1\cdots t_u;t_1'\cdots t_v')\}^{\frac{1}{2}}$$

is dependent only on u, $(1 \leqq u \leqq \ell-1)$.

Consider now a balanced array of strength t, m constraints and index set $\Lambda_t = \{\mu_i;i=0,1,\cdots,t\}$ which may yield an AB or APB design. See, for example, Shirakura [4] for its definition. The above array is denoted simply by B-array$[m, t; \Lambda_t]$. In particular, a B-array$[m, t; \Lambda_t]$ is called an orthogonal array of strength t, m constraints and index λ (denoted simply by O-array$[m, t; \lambda]$) if $\mu_0 = \mu_1 = \cdots = \mu_t$ $(= \lambda$, say).

Theorem 3.1. *If a fraction T is an O-array$[m, 2\ell-1; \lambda]$, then T is an AB design.*

Proof. For an O-array$[m, 2\ell-1; \lambda]$, its information matrix M can be expressed in the form $M = \mathrm{diag}(bI_{p+1}, F)$, where $b = 2^{2\ell-1}\lambda$ and F is some

$\binom{m}{\ell} \times \binom{m}{\ell}$ matrix (see [9]). Hence it is easy to see that a matrix X satisfying (1.3) is given by $X = [\underline{0}: b^{-1}I_p : O]$. Therefore $X_1 = CX' = b^{-1}I_p$. Since the diagonal elements of XHX' in (2.1) are equal to $\sum_{\alpha=2}^{s} \alpha(\alpha-1) \cdot b^{-2}N_\alpha$, those of AA' are equal to $2^m(b^{-1} + \sum_{\alpha=2}^{s}\alpha(\alpha-1)b^{-2}N_\alpha) - 1$, which completes the proof.

Recall an $(\ell+1)$ sets TMDPB association algebra $\mathcal{A}$ generated by some $\nu \times \nu$ symmetric matrices which has been introduced in [9]. From the properties of $\mathcal{A}$, we can easily obtain

Lemma 3.2. *For the alias matrix A of a fraction T, if the $\nu \times \nu$ matrix* $\mathrm{diag}(0, AA', O)$ *belongs to $\mathcal{A}$, then T is an APB design.*

Theorem 3.3. *If a fraction T is a B-array$[m, 2\ell; \Lambda_{2\ell}]$ whose columns are all distinct, then T is an APB design.*

Proof. It has been shown in [5] that the matrix X_1 in (1.4) satisfies $\mathrm{diag}(0, X_1, O) \in \mathcal{A}$. By Lemma 2.1, therefore, $\mathrm{diag}(0, AA', O) \in \mathcal{A}$, since H of (2.2) vanishes. This completes the proof, because of Lemma 3.2.

Theorem 3.4. *Let T be a B-array$[m, 2\ell; \Lambda_{2\ell}]$ as a fraction. If $E_\alpha' E_\alpha \in \mathcal{A}$, where E_α is the design matrix for $\underline{\theta}$ of $T_\alpha (\neq \phi)$, then T is an APB design.*

Proof. From [5], it is also seen that for a B-array$[m, 2\ell; \Lambda_{2\ell}]$ T, a matrix X in (1.3) can be determined to satisfy $[\underline{0} : X' : O]$ $(= X^\circ$, say$) \in \mathcal{A}$. Since H of (2.2) belongs to $\mathcal{A}$, $X^{\circ\prime} H X^\circ = \mathrm{diag}(0, XHX', O)$ also belongs to $\mathcal{A}$. Hence it follows from (2.1) that $\mathrm{diag}(0, AA', O)$ belongs to $\mathcal{A}$. This completes the proof, because of Lemma 3.2.

Corollary 3.5. *Let* T^* *be the balanced array of Theorem* 3.3. *Let* T *be a fraction obtained by replicating* T^* r (≥ 2) *times (i.e.,* $r_1 = r_2 = \cdots = r_n = r$*). Then* T *is an APB design.*

Proof. Clearly, T is a balanced array of strength 2ℓ. By the assumption, $s = r$ and $T_\alpha = \phi$ for $\alpha = 1,\cdots,s-1$. Also E_s is the design matrix of T^*. This means $E_s' E_s \in A$, which completes the proof.

Let $\underline{1}$ be the $m \times 1$ vector whose elements are all one.

Corollary 3.6. *For integers* $\zeta_1 \geq 2$ *and* $\zeta_2 \geq 2$*, let*

$$T = [\underbrace{\underline{0} : \underline{0} : \cdots : \underline{0}}_{\zeta_1} : \underbrace{\underline{1} : \underline{1} : \cdots : \underline{1}}_{\zeta_2} : T^*],$$

where T^* *is a B-array*$[m,\ 2\ell;\ \Lambda^*_{2\ell} = \{\mu_i^*\}]$ *whose columns are all distinct and exclusive of* $\underline{0}$ *and* $\underline{1}$. *Then* T *is an APB design.*

Proof. It is clear that T is a B-array$[m,\ 2\ell;\ \Lambda_{2\ell}]$ with $\mu_0 = \mu_0^* + \zeta_1$, $\mu_{2\ell} = \mu^*_{2\ell} + \zeta_2$ and $\mu_i = \mu_i^*$ $(i = 1,\cdots,2\ell-1)$. From (1.1), the $\nu \times 1$ coefficient vectors $\underline{e}_1$ and $\underline{e}_2$ for $\underline{\theta}$ according to the assemblies $\underline{0}'$ and $\underline{1}'$, respectively, are given by

$$\underline{e}_1' = (1;\ -1,\ -1,\cdots,-1;\ 1,\ 1,\cdots,\ 1;\ \cdots\ ;\ (-1)^\ell,\ (-1)^\ell,\cdots,\ (-1)^\ell),$$

$$\underline{e}_2' = (1;\ \underbrace{1,\ 1,\cdots,\ 1}_{m};\ \underbrace{1,\ 1,\cdots,\ 1}_{\binom{m}{2}};\ \cdots\ ;\ \underbrace{1\ ,\ \ 1\ ,\cdots,\ \ 1}_{\binom{m}{\ell}}\ ).$$

This implies that $\underline{e}_1\underline{e}_1' \in A$ and $\underline{e}_2\underline{e}_2' \in A$ (see [9]). Therefore if $\zeta_1 = \zeta_2$, then $E'_{\zeta_1}E_{\zeta_1} = \underline{e}_1\underline{e}_1' + \underline{e}_2\underline{e}_2' \in A$. If $\zeta_1 \neq \zeta_2$, then $E'_{\zeta_1}E_{\zeta_1} = \underline{e}_1\underline{e}_1' \in A$ and $E'_{\zeta_2}E_{\zeta_2} = \underline{e}_2\underline{e}_2 \in A$. This shows that T is an APB design.

Note that for each pair (ζ_1, ζ_2) with $(\zeta_1 \geqq 2; \zeta_2 = 0, 1)$ and $(\zeta_1 = 0, 1; \zeta_2 \geqq 2)$, it can be shown similarly that T in (3.1) is also an APB design. See also Theorem 3.3 for $\zeta_1, \zeta_2 = 0, 1$.

Consider now a simple array with parameters $[m; \lambda_0, \lambda_1, \cdots, \lambda_m]$ as a fraction (see, e.g., [4]). This array is a special case of a balanced array. For practical values of m and N, however, most of balanced arrays of strength 4 or 6 are simple arrays.

Corollary 3.7. *Let T be a simple array with parameters $[m; \lambda_0, \lambda_1, \cdots, \lambda_m]$. Then T is an APB design.*

Proof. By the definition of a simple array, it can easily be shown that $E_\alpha' E_\alpha \in A$ for every α with $1 \leqq \alpha \leqq s$ and $T_\alpha \neq \phi$. This completes the proof.

As noted in Section 1, we assume a fraction T to be a design of resolution 2ℓ. Designs of resolution $2\ell+1$ in which $\underline{\theta}$ is estimable ignoring $\underline{\theta}^*$, of course, are also of resolution 2ℓ. A design of resolution $2\ell+1$ is equivalent to the non-singularity of the information matrix. As seen from (1.3), however, it is in general difficult to verify that a fraction is a design of resolution 2ℓ which is not of resolution $2\ell+1$. For a B-array $[m, 2\ell; \Lambda_{2\ell}]$ T such that the information matrix M is singular, Shirakura [2, 4] has given sufficient conditions for T to be a 2^m-FF design of resolution 2ℓ. Subsequently, Shirakura [5] has given a necessary and sufficient condition for this.

4. AB and APB designs generated by level permutations. Consider a set $\Omega = \{\underline{\omega}' = (\omega_1, \omega_2, \cdots, \omega_m) ; \omega_i = 0 \text{ or } 1, i = 1,2,\cdots,m\}$. For a fraction

T with N assemblies and any $\underline{\omega}$ in Ω, define

$$T(\underline{\omega}) = T + J(\underline{\omega}) \;, \quad (\text{mod } 2),$$

where $J(\underline{\omega})$ denotes the $m \times N$ matrix with all columns $\underline{\omega}$. An element $\underline{\omega}$ in Ω is called a level permutation and $T(\underline{\omega})$ is called a generated fraction by $\underline{\omega}$. The following lemma has been established in [8]:

Lemma 4.1. *Let* T *and* $T(\underline{\omega})$ *be a fraction and its generated fraction, respectively. Let* $E(\underline{\omega})$ *and* $E^*(\underline{\omega})$ *be the design matrices for* $\underline{\theta}$ *and* $\underline{\theta}^*$ *of* $T(\underline{\omega})$*, respectively. Then*

$$E(\underline{\omega}) = E\,D(\underline{\omega}) \quad \text{and} \quad E^*(\underline{\omega}) = E^*P^* \tag{4.1}$$

hold where P^* *is some* $\nu^* \times \nu^*$ *orthogonal matrix and* $D(\underline{\omega})$ *is the* $\nu \times \nu$ *matrix given by*

$$D(\underline{\omega}) = \text{diag}(1;\ (-1)^{\omega_1}, \cdots, (-1)^{\omega_m};\ (-1)^{\omega_1+\omega_2}, \cdots, (-1)^{\omega_{m-1}+\omega_m};$$
$$\cdots\ ;\ (-1)^{\omega_1+\cdots+\omega_\ell}, \cdots, (-1)^{\omega_{m-\ell+1}+\cdots+\omega_m}).$$

Theorem 4.2. *If* T *is an AB (or APB) design, then for every* $\underline{\omega}$ *in* Ω*,* $T(\underline{\omega})$ *is also an AB (or APB) design.*

Proof. Let $M(\underline{\omega})$ be the information matrix of $T(\underline{\omega})$. Then by Lemma 4.1 we have $M(\underline{\omega}) = D(\underline{\omega})\,M\,D(\underline{\omega})$. From (1.3), it is easy to see that

$$D_0(\underline{\omega})\,X\,D(\underline{\omega})\,M(\underline{\omega}) = D_0(\underline{\omega})\,C\,D(\underline{\omega}) = C \;,$$

where $D_0(\underline{\omega}) = C\,D(\underline{\omega})\,C'$. Define $X(\underline{\omega}) = D_0(\underline{\omega})\,X\,D(\underline{\omega})$. Then $X(\underline{\omega})\,M(\underline{\omega}) = C$. This implies that $T(\underline{\omega})$ is also a design of resolution 2ℓ. Moreover,

$$A(\underline{\omega}) = X(\underline{\omega})\,E(\underline{\omega})'\,E^*(\underline{\omega}) = D_0(\underline{\omega})\,X\,E'\,E^*P^* \;,$$

where $A(\underline{\omega})$ is the alias matrix of $T(\underline{\omega})$. By Lemma 4.1, it is clear that

$$A(\underline{\omega})\,A(\underline{\omega})' = D_0(\underline{\omega})\,A\,A'\,D_0(\underline{\omega}).$$

This shows that the diagonal elements of AA' are invariant under a level permutation. The proof is completed.

It is easy to verify that if T is an O-array$[m, 2\ell-1; \lambda]$, then $T(\underline{\omega})$ is also an O-array$[m, 2\ell-1; \lambda]$. In this case, the result of Theorem 4.2 has been mentioned in Theorem 3.1. However, for a balanced array (or simple array) T of Theorem 3.3 and Corollaries 3.5 - 3.7, $T(\underline{\omega})$ can not be a balanced array of strength 2ℓ for every $\underline{\omega}$ in $\Omega - \{\underline{0}, \underline{1}\}$ as long as T is neither an O-array$[m, 2\ell; \lambda']$ nor O-array$[m, 2\ell-1; \lambda]$, (see [6]). This implies that $(2^m - 2)$ distinct APB designs which are not balanced arrays of strength 2ℓ can be generated from such an array T by level permutations. For any two fractions T_1 and T_2 with N assemblies, T_1 is said to be distinct from T_2 if $T_1 \neq T_2$ as sets of assemblies. Finally, note that for a B-array$[m, 2\ell; \Lambda_{2\ell}]$ T, $T(\underline{1})$ is also a B-array$[m, 2\ell; \tilde{\Lambda}_{2\ell} = \{\tilde{\mu}_i = \mu_{2\ell-i}\ ;\ i = 0, \cdots, 2\ell\}]$.

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