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Modularity in Semilattices

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1. Introduction. It is customary to call an upper semilattice $(S; \vee)$ *distributive (modular)* if, whenever $x \leq y \vee z$ (and $z \leq x$), there exist $y_1 \leq y$ and $z_1 \leq z$ (there exists $y_1 \leq y$) such that $x = y_1 \vee z_1$ ($x = y_1 \vee z$). A distributive semilattice is modular and, most importantly from the point of view of this note, a modular semilattice is downwards directed. Moreover, amongst downwards directed semilattices, the distributive and modular semilattices are precisely those, whose lattices of non-empty ideals are respectively distributive and modular, in the usual terminology of Lattice Theory. Such semilattices have been studied intensively; see the papers in the References and [2], in particular, for many characterizations.

In [4; Definition 4], Katriňák considered a generalization of distributivity which applies to *all* upper semilattices. Our purpose is to give a corresponding generalization of modularity.

2. Generalized modular semilattices. An upper semilattice S is said to be *generalized distributive* if, whenever $x, y, z \in S$ are such that $x \leq y \vee z$, $x \not\leq y$, and $x \not\leq z$, there exist $y_1 \leq y$ and $z_1 \leq z$ such that $x = y_1 \vee z_1$. In [4], Katriňák uses the term "distributive" in place of our "generalized distributive"; we reserve the former term for those semilattices mentioned in the Introduction. Our generalization of modularity is

a first order condition having a somewhat different appearance.

An upper semilattice S is said to be *generalized modular* if, whenever $x, y, z \in S$ are such that $x \leq y \vee z$ and $z < x$, then there exists $y_1 \leq y$ such that $x = y_1 \vee z$.

In [4; 1.5], Katriňák showed a semilattice is distributive if and only if it is both generalized distributive and downwards directed. Correspondingly, we have:

Proposition 2.1. *An upper semilattice S is modular if and only if it is both generalized modular and downwards directed.*

Proof. Suppose $z \leq x \leq y \vee z$ in a downwards directed, generalized modular semilattice S . If $z < x$ then, by generalized modularity, $x = y_1 \vee z$ for some $y_1 \leq y$. If $z = x$ then use the downwards directedness of S to obtain y_1 such that $y_1 \leq y, x$. Thus, $x = y_1 \vee x = y_1 \vee z$ and $y_1 \leq y$. Hence, S is modular.

A modular semilattice is well known to be directed below, see, for example, [1; Proposition 1.1]. Hence, the converse holds. $\square$

An *ideal* of an upper semilattice S is a *non-empty* subset K such that $z \in K$ whenever $z \leq x$ and $x \in K$, and $a \vee b \in K$ whenever $a, b \in K$. Ordered by set-inclusion, the set $J(S)$ of all ideals of S forms an upper semilattice wherein the supremum $K_1 \vee K_2$ of two ideals K_1 and K_2 is given by $K_1 \vee K_2 = \{x \in S: x \leq y \vee z \text{ for some } y \in K_1 \text{ and } z \in K_2\}$. As is well known and easy to see, the following conditions are equivalent:

(1) S is downwards directed; (2) $J(S)$ is a lattice wherein the infimum is set-intersection; (3) $J(S)$ is a lattice. In order to overcome

the above restrictions, we follow Katriňák by introducing the set $J_0(S) = J(S) \cup \{\emptyset\}$. In other words, we admit the empty set, $\emptyset$, as an ideal and refer to it as *the empty ideal*. Ordered by set-inclusion, $J_0(S)$ is always a lattice and $J(S)$ is an upper subsemilattice of it.

In [4; 1.6], Katriňák proved that an upper semilattice S is generalized distributive if and only if $J_0(S)$ is a distributive lattice. The related result is:

Theorem 2.2. *An upper semilattice S is generalized modular if and only if $J_0(S)$ is a modular lattice.*

Proof. Suppose S is generalized modular. Let $I, J, K \in J_0(S)$. Consider $I \cap ((I \cap J) \vee K)$. If this is the empty ideal then $I \cap J = \emptyset = I \cap K$ and so $I \cap ((I \cap J) \vee K) = (I \cap J) \vee (I \cap K)$, as required. If $I \cap J = \emptyset$ then $I \cap ((I \cap J) \vee K) = I \cap (\emptyset \vee K) = I \cap K = (I \cap J) \vee (I \cap K)$, as required. If $K = \emptyset$ then $I \cap ((I \cap J) \vee K) = I \cap (I \cap J) = I \cap J = (I \cap J) \vee (I \cap K)$, as required. Thus, we can assume that $I \cap ((I \cap J) \vee K)$, $I \cap J$ and K are all non-empty. Let $x \in I \cap ((I \cap J) \vee K)$ so that $x \in I$ and $x \leq y \vee z$ for some $y \in I \cap J$ and $z \in K$. Then, $y \leq y \vee x \leq y \vee z$. If $y = y \vee x$ then $x \leq y$ and so $x \in I \cap J$, and consequently $x \in (I \cap J) \vee (I \cap K)$. If $y < y \vee x$ then, by generalized modularity $y \vee x = y \vee z_1$ for some $z_1 \leq z$. But $y \vee x \in I$ and $z_1 \leq y \vee x$. Hence $z_1 \in I \cap K$. Then, $x \leq y \vee z_1$ implies that $x \in (I \cap J) \vee (I \cap K)$. Once again, it follows that $I \cap ((I \cap J) \vee K) = (I \cap J) \vee (I \cap K)$.

Conversely, suppose $J_0(S)$ is a modular lattice. Let $z < x \leq y \vee z$. For $t \in S$, let $(t) = \{r \in S: r \leq t\}$ be the principal ideal generated by t . Then, $(z) \subset (x) \subset (y \vee z) = (y) \vee (z)$. Using the modularity of $J_0(S)$, we

obtain $[x] = [x] \cap ([y] \vee [z]) = ([x] \cap [y]) \vee [z]$. But $[x] \cap [y] \neq \emptyset$, as otherwise $[x] = \emptyset \vee [z] = [z]$ and $x = z$. Hence, there exists $y_1 \in [x] \cap [y]$ such that $x \leq y_1 \vee z$ for some $z_1 \in [z]$. Of course, $y_1 \vee z \leq x$ and so $x = y_1 \vee z$, as required. $\square$

It is an immediate corollary of 2.2 and [4; 1.6] that generalized distributivity implies generalized modularity. However, this also follows from a simple direct argument.

3. Ordinal sums. With Theorem 2.2, we have plugged a gap in the literature. We now turn to the significance of generalized modular and distributive semilattices.

Let S and T be two partially ordered sets. The ordinal sum $S \oplus T$ of S and T is their disjoint union $S \dot{\cup} T$, together with the partial order defined by: for $x, y \in S \dot{\cup} T$, $x \leq y$ if and only if either $x, y \in S$ and $x \leq y$ in S , or $x, y \in T$ and $x \leq y$ in T , or $x \in S$ and $y \in T$. The ordinal sum is a partially ordered set and it is an upper semilattice if and only if both S and T are upper semilattices.

Theorem 3.1. *Let S and T be two upper semilattices and $R = S \oplus T$ be their ordinal sum. Then, R is generalized modular (generalized distributive) if and only if both S and T are generalized modular (generalized distributive).*

Proof. Suppose R is generalized modular. Since S is an ideal in R , it is clear that S is generalized modular. Now let $x, y, z \in T$ be such that $z < x \leq y \vee z$. Let $y_1 \in R$ be such that $y_1 \leq y$ and $x = y_1 \vee z$. Now $y_1 \notin S$; otherwise $y_1 \in S$ and so $y_1 < z$ and $x = z$, an impossibility.

Thus, $y_1 \in T$ and T is generalized modular.

Suppose S and T are generalized modular. Let $x, y, z \in R$ be such that $z < x \leq y \vee z$. If $z \in T$ then $x \in T$ and in this event $y \notin S$ since otherwise $y \in S$, $y < z$ and $z < x \leq z$. Hence, $z \in T$ implies $x, y \in T$ and so there exists $y_1 \in T$ such that $y_1 \leq y$ and $x = y_1 \vee z$. Suppose $z \in S$ and $x \in T$. As $x \leq y \vee z$, $y \in T$. Thus, $y \vee z = y$ and $x = y_1 \vee z$, with $y_1 = x \leq y$. Suppose $z \in S$ and $x \in S$. If $y \in S$ then, because S is generalized modular, there is nothing to prove. Consequently, suppose $y \in T$. Hence, $x < y$. Then, $x = y_1 \vee z$, where $y_1 = x \leq y$. Hence, R is generalized modular.

A similar argument holds in the generalized distributive case; we omit details. $\square$

Corollary 3.2. *Let S and T be upper semilattices and $R = S \oplus T$ be their ordinal sum. Then, R is a modular (distributive) if and only if S is a modular (distributive) and T is a generalized modular (generalized distributive).*

Proof. R is directed below if and only if S is directed below. $\square$

Corollary 3.3. *Let L be a lattice with a smallest element 0 . Let $S = L \setminus \{0\}$ be endowed with the partial order inherited from L , so that S is an upper semilattice. Then, S is generalized modular (generalized distributive) if and only if L is a modular (distributive) lattice.*

Proof. L is as an upper semilattice, the ordinal sum $\{0\} \oplus S$. Moreover, L is a modular (distributive) lattice if and only if its underlying upper semilattice is modular (distributive). $\square$

Corollary 3.4. *Let S be an upper semilattice. Then, S is generalized modular (generalized distributive) if and only if $J(S)$ is a generalized modular (generalized distributive) upper semilattice.*

Proof. $J_0(S)$ is always a lattice and the ordinal sum $\{\emptyset\} \oplus J(S)$, where $\emptyset$ is the smallest element of $J_0(S)$ and $J(S)$ is the upper semilattice $J_0(S) \setminus \{\emptyset\}$. Thus, the result follows from 3.3, 2.2 and [4; 1.6]. $\square$

Corollary 3.5. *Let S be a finite semilattice. Then, S is generalized modular (generalized distributive) if and only if the ordinal sum $\{0\} \oplus S$, $0 \notin S$, is a modular (distributive) lattice.*

Proof. If S is generalized modular, $\{0\} \oplus S$ is a finite modular semilattice. Also, a directed below, finite semilattice is a lattice. $\square$

Thus, Theorem 3.1 is quite fertile. Corollary 3.5 enables us to describe all finite generalized modular (generalized distributive) semilattices. In particular, the *dual trees* or *root systems* which consist of a largest element 1 and a finite number, $n \geq 2$, of incomparable elements whose suprema are all 1, are all generalized modular and generalized distributive if and only if $n \geq 3$ and $n = 2$, respectively. By forming the ordinal sum $N \oplus T_n$ of the upper semilattice N of natural numbers, endowed with the usual order, with the above type of dual tree T_n , having n incomparable elements, we obtain a modular or distributive semilattice according to whether $n \geq 3$ or not, due to Corollary 2.2. This type of construction has appeared in the literature frequently; in [4; Fig. 2, p.134] Katriňák mentioned the case where $n = 2$ as a distributive semilattice, while in [1; p.241] and [3; p.67] the author, and Hofmann,

Mislove and Stralka, respectively, cite the case where $n = 3$ as an example of a modular but not distributive semilattice. Also, on the basis of 3.2 and 3.5, we can easily find a rationalization for $N \oplus P$, with N as above and P equal to the four element dual tree obtained from the five element non-modular lattice by eliminating its smallest element, being the "simplest" downwards directed upper semilattice which is not modular.

Because of the literature, it seems that modular and distributive semilattices are interesting objects and so, 3.2 says that generalized modular and distributive semilattices have some interest. However, in the next section, we show that these latter classes a damning feature.

4. Retractions. Recall that a subsemilattice T of an upper semilattice S is a retract of S , and $f: S \rightarrow T$ is a retraction, if there is a function f mapping S onto T such that the restriction of f to T is identity on T and, of course, f preserves the supremum of a pair of elements of S .

The main theorems of Rhodes' paper [5] give the generalizations of the famous Dedekind and Birkhoff criteria for modularity and distributivity of a lattice in terms of semilattice retractions instead of sublattices. In [2; Theorems 3.4, 4.2, 4.4 and their corollaries], the author used Rhodes' Theorems to obtain apparently difficult results. For non-downwards directed semilattices, the appropriate modifications of Rhodes' Theorems are:

Theorem 4.1. Let S be an upper semilattice. Then,

(i) S is generalized modular if and only if neither the upper semilattice which is the five element non-modular lattice nor its four element subsemilattice, obtained by deleting the smallest element, is isomorphic,

as an upper subsemilattice, to a retract of S ,

(ii) S is generalized modular but not generalized distributive if and only if the five element modular non-distributive lattice nor its four element subsemilattice, obtained by deleting the smallest element, is isomorphic, as an upper subsemilattice, to a retract of S .

Proof. Adjoin a new smallest element to S and apply Rhodes' Theorems in the light of 3.2. It should be noted that the proper semilattices mentioned in (i) and (ii) are non-modular and non-distributive respectively, c.f. 3.5 and the fact generalized modularity and distributivity are preserved under retractions. $\square$

It should be mentioned that both possibilities in (i) and (ii) of 4.1 can occur. For example, let S be the non-modular non-lattice obtained from the five element non-modular lattice by adjoining a new element whose supremum with all elements, except itself, is the largest element of the original lattice. Then, this semilattice is not downwards directed, yet it has the original lattice as a retract.

Finally, let $I, K \in J(S)$, S being an upper semilattice. Define a semilattice congruence $\Phi(I)$ by: $x \equiv y (\Phi(S))$ if and only if $x, y \in I$ or $x, y \in S \setminus I$. Then, $S/\Phi(I)$ is isomorphic to the two-chain, 2, provided $S \setminus I \neq \emptyset$, and the two-chain becomes a semilattice retract of S . If $I \cap K = \emptyset$ then $\Phi(I) \cap \Phi(K)$ has I, K and $S \setminus (I \cup K)$ as distinct congruence classes and the three element semilattice, Λ , obtained by deleting the smallest element from the four element Boolean lattice becomes a retract of S . Of course, $\Lambda \times 2$ is not generalized modular. Hence, we are led to a result, which was pointed out to me by my student Robert C. Hickman.

Proposition 4.2. *Let S and T be upper semilattices such that S and T have at least two elements. Then, the semilattice direct product $S \times T$ is downwards directed if and only if $\Lambda \times 2$ is not a retract of $S \times T$. Consequently, $S \times T$ is generalized modular (generalized distributive) if and only if both S and T are modular (distributive), and in which case $S \times T$ is actually modular (distributive).*

Proof. The first statement follows from our preliminary comments and the fact that the direct product of retracts is itself a retract, while the second is routinely verifiable. □

In conclusion, what can be said about categorical products of non-downwards directed generalized modular semilattices?

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