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Filipoiu, Alexandru

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Analytic Tableaux for θ -Valued Propositional Logic

By Alexandru FILIPOIU

The θ -valued Lukasiewicz algebras were introduced by Professor Gr. C. Moisil in 1968 [3] (see also [2]). In [1] the θ -valued Propositional Logic was formulated. The purpose of this paper is to give an application of the method of Analytic Tableaux (see [4]) to this logic.

§ 1. Formulas of θ -valued propositional logic. Let J be a chain with a first element (0) and a last element (1), with the order type θ . We shall use the following symbols:

- (i) An infinite set V of symbols, called propositional variables.
- (ii) The logical connectives: $\vee$, $\wedge$, ϕ_α , $\overline{\phi}_\alpha$ for every $\alpha \in J$.
- (iii) The two symbols $(,)$ respectively called the left parenthesis and the right parenthesis (they are used for purposes of punctuation).

Definition 1.1. By a *formation sequence*, we mean any finite sequence such that each term of the sequence is either a propositional variable or of the forms $(A \vee B)$, $(A \wedge B)$, where A, B are earlier terms of the sequence, or of the forms $\phi_\alpha(A)$, $\overline{\phi}_\alpha(A)$ for some $\alpha \in J$, where A is an earlier term of the sequence. We define A to be a *formula* if there exists a formation sequence whose last term is A , and such a sequence is also called a *formation sequence for A* .

We shall use the letters A, B, X to denote formulas. It can be proved that every formula can be formed in only one way, i.e. for every formula X , one and only one of the following conditions holds:

- (1) X is a propositional variable,
- (2) There is a unique pair (X_1, X_2) such that $X = (X_1 \vee X_2)$ or $X = (X_1 \wedge X_2)$.
- (3) There is a unique formula Y and a unique $\alpha \in J$, such that $X = \phi_\alpha(Y)$ or $X = \bar{\phi}_\alpha(Y)$.

The notion of immediate subformula, subformula and of formation tree is given as in the case of propositional logic in [4].

§ 2. θ -valued valuations and truth-sets. We shall consider the θ -valued Lukasiewicz algebra $DL_2 = \{f: J \rightarrow L_2 \mid \alpha \leq \beta \text{ implies } f(\alpha) \geq f(\beta)\}$ with the operations given by:

$$(f \cup g)(\alpha) = f(\alpha) \cup g(\alpha), \quad (f \cap g)(\alpha) = f(\alpha) \cap g(\alpha),$$

$$(\mathcal{F}_\alpha f)(\beta) = f(\alpha), \quad (\bar{\mathcal{F}}_\alpha f)(\beta) = \overline{f(\alpha)}.$$

Definition 2.1. A function $v: F \rightarrow DL_2$, where F is the set of formulas, is called a θ -valued valuation if for every $X, Y \in F$ and $\alpha \in J$, the following conditions hold:

- a) $v(X \vee Y) = v(X) \cup v(Y)$, $v(X \wedge Y) = v(X) \cap v(Y)$
- b) $v[\phi_\alpha(X)] = \mathcal{F}_\alpha v(X)$, $v[\bar{\phi}_\alpha(X)] = \bar{\mathcal{F}}_\alpha v(X)$

It can be easily proved that every function $v_0: V \rightarrow DL_2$, can be extended in exactly one way to a θ -valued valuation of F .

Definition 2.2. a) $X \in F$ is an α -tautology ($\alpha \in J$) iff for every θ -valued valuation of F , $v(X)(\alpha) = 1$.

- b) $S \subset F$ is called α -satisfiable if there is at least one θ -valued val-

uation v such that $v(X)(\alpha) = 1$ for every $X \in S$.

c) A set $S \subset F$ α -implies a formula X if $v(X)(\alpha) = 1$ for every θ -valued valuation v which α -satisfies S .

If $v: F \rightarrow DL_2$ is a θ -valued valuation and $V_\alpha = \{X \in F \mid v(X)(\alpha) = 1\}$ we have:

$$\begin{aligned} X \wedge Y \in V_\alpha & \text{ iff } X, Y \in V_\alpha, \\ X \vee Y \in V_\alpha & \text{ iff } X \text{ or } Y \in V_\alpha, \\ \phi_\beta(X) \in V_\alpha & \text{ iff } X \in V_\beta, \\ \overline{\phi}_\beta(X) \in V_\alpha & \text{ iff } \phi_\beta(X) \notin V_\alpha, \end{aligned}$$

if $\alpha \leq \beta$, $X \in V_\beta$ then $X \in V_\alpha$.

Definition 2.3. A family $\{V_\alpha \mid \alpha \in J\}$, $V_\alpha \subset F$, $\alpha \in J$ will be called *saturated* (or *truth-set*) if the following conditions hold:

- 1) for every $\alpha \in J$, $X \wedge Y \in V_\alpha$ iff X and $Y \in V_\alpha$.
- 2) for every $\alpha \in J$, $X \vee Y \in V_\alpha$ iff X or $Y \in V_\alpha$.
- 3) for every β , $\alpha \in J$, $\phi_\beta(X) \in V_\alpha$ iff $X \in V_\beta$.
- 4) for every α , $\beta \in J$, $\overline{\phi}_\beta(X) \in V_\alpha$ iff $\phi_\beta(X) \notin V_\alpha$.
- 5) for every $\alpha \leq \beta \in J$, $V_\alpha \supset V_\beta$.

Remark 1. If $\text{Taut}_\alpha = \{X \in F \mid X \text{ } \alpha\text{-tautology}\}$ then $\text{Taut}_\alpha = \bigcap \{V_\alpha \mid v \text{ } \theta\text{-valued valuation}\}$

Remark 2. If $v: F \rightarrow DL_2$ is a function and $V_\alpha = \{X \in F \mid v(X)(\alpha) = 1\}$ then the following conditions are equivalent:

- a) v is a θ -valued valuation.
- b) $\{V_\alpha \mid \alpha \in J\}$ is a saturated family.

Remark 3. If we have a family $\{V_\alpha \mid \alpha \in J\}$, $V_\alpha \subset F$ and we define $v: F$

$\rightarrow L_2^J$ by $v(X)(\alpha) = 1$ iff $X \in V_\alpha$, then the following conditions are equivalent:

- a) $\{V_\alpha \mid \alpha \in J\}$ is a saturated family.
- b) v is a θ -valued valuation.

§ 3. The method of analytic tableaux. We now describe a proof procedure for θ -valued propositional logic. We introduce the symbols T^α , $\bar{T}^\alpha$ for every $\alpha \in J$ to our object language, and define a signed formula as an expression $T^\alpha(X)$ or $\bar{T}^\alpha(X)$, where $X \in F$.

Rules for the construction of tableaux. For every $\alpha \in J$ and each logical connective there are two rules; one of which concerns with a formula preceded by T^α and the other concerns with a formula preceded by $\bar{T}^\alpha$.

$$\begin{array}{ll}
 1) \frac{T^\alpha((A \vee B))}{T^\alpha(A) \mid T^\alpha(B)} & \frac{\bar{T}^\alpha((A \vee B))}{\frac{\bar{T}^\alpha(A)}{\bar{T}^\alpha(B)}} \quad 3) \frac{T^\alpha(\phi_\beta(A))}{T^\beta(A)} \quad \frac{\bar{T}^\alpha(\phi_\beta(A))}{\bar{T}^\beta(A)} \\
 2) \frac{T^\alpha((A \wedge B))}{\frac{T^\alpha(A)}{T^\alpha(B)}} & \frac{\bar{T}^\alpha((A \wedge B))}{\bar{T}^\alpha(A) \mid \bar{T}^\alpha(B)} \quad 4) \frac{T^\alpha(\bar{\phi}_\beta(A))}{\bar{T}^\beta(A)} \quad \frac{\bar{T}^\alpha(\bar{\phi}_\beta(A))}{\bar{T}^\beta(A)}
 \end{array}$$

Signed formulas, other than signed variables, are of two types:

(A) those which have direct consequences $(\bar{T}^\alpha((A \vee B)), T^\alpha((A \wedge B)), T^\alpha(\phi_\beta(A)), \bar{T}^\alpha(\phi_\beta(A)), T^\alpha(\bar{\phi}_\beta(A)), \bar{T}^\alpha(\bar{\phi}_\beta(A)))$; (B) those which branch $(T^\alpha((A \vee B)), \bar{T}^\alpha((A \wedge B)))$. We use the letter Ω to stand for any signed formula of type (A), and the letter Γ to stand for any signed formula of type (B); all our tableaux rules can be succinctly lumped into the following two:

$$\text{Rule A: } \frac{\Omega}{\frac{\Omega_1}{\Omega_2}} \quad \text{Rule B: } \frac{\Gamma}{\Gamma_1 \mid \Gamma_2}$$

where we define Ω_1 , Ω_2 , Γ_1 and Γ_2 as follows:

$$\text{if } \Omega = T^\alpha(X \wedge Y) \text{ then } \Omega_1 = T^\alpha(X) \text{ and } \Omega_2 = T^\alpha(Y);$$

- if $\Omega = \overline{T}^\alpha(X \vee Y)$ then $\Omega_1 = \overline{T}^\alpha(X)$ and $\Omega_2 = \overline{T}^\alpha(Y)$;
 if $\Omega = T^\alpha \phi_\beta(X)$ then $\Omega_1 = \Omega_2 = T^\beta(X)$;
 if $\Omega = T^\alpha \overline{\phi}_\beta(X)$ then $\Omega_1 = \Omega_2 = \overline{T}^\beta(X)$;
 if $\Omega = \overline{T}^\alpha \phi_\beta(X)$ then $\Omega_1 = \Omega_2 = \overline{T}^\beta(X)$;
 if $\Omega = \overline{T}^\alpha \overline{\phi}_\beta(X)$ then $\Omega_1 = \Omega_2 = T^\beta(X)$;
 if $\Gamma = \overline{T}^\alpha(X \wedge Y)$ then $\Gamma_1 = \overline{T}^\alpha(X)$ and $\Gamma_2 = \overline{T}^\alpha(Y)$;
 if $\Gamma = T^\alpha(X \vee Y)$ then $\Gamma_1 = T^\alpha(X)$ and $\Gamma_2 = T^\alpha(Y)$.

Lemma. Let S be a set of signed formulas which is of the form $S = (\bigcup_{\alpha \in J} S^\alpha) \cup (\bigcup_{\alpha \in J} \overline{S}^\alpha)$, where

$$S^\alpha = \{X \in S \mid X = T^\alpha(Y) \text{ for some } Y \in F\},$$

$$\overline{S}^\alpha = \{X \in S \mid X = \overline{T}^\alpha(Y) \text{ for some } Y \in F\}$$

such that for every $\alpha \leq \beta \in J$, we have $S_\alpha^* \supset S_\beta^*$ and $\overline{S}_\alpha^* \subset \overline{S}_\beta^*$, where $S_\alpha^* = \{X \in F \mid T^\alpha(X) \in S^\alpha\}$ and $\overline{S}_\alpha^* = \{X \in F \mid \overline{T}^\alpha(X) \in \overline{S}^\alpha\}$. Then $\{S_\alpha^* \mid \alpha \in J\}$ is a saturated family iff the following conditions hold:

- a) for every $\alpha \in J$ and every $X \in F$, exactly one of $T^\alpha(X)$, $\overline{T}^\alpha(X)$ belongs to S .
- b) $\Omega \in S$ iff Ω_1 and $\Omega_2 \in S$
- c) $\Gamma \in S$ iff Γ_1 or $\Gamma_2 \in S$.

Proof. If we define $v(X)(\alpha) = 1$ iff $T^\alpha(X) \in S^\alpha$ and $v(X)(\alpha) = 0$ iff $\overline{T}^\alpha(X) \in \overline{S}^\alpha$, it is immediate that $v: F \rightarrow 2^J$ is a 0-valued valuation iff the conditions a), b) and c) hold.

Definition 3.1. A set S of signed formulas is called *saturated* if it satisfies the conditions a), b), c) of the lemma.

Definition 3.2. An analytic tableau for a signed formula X is an or-

dered dyadic tree (see [4]), whose points are formulas, which is constructed as follows: We start by placing X at the origin. Now suppose T is a tableau for X which has already been constructed; let Y be an end point. Then we may extend T by either of the following two operations:

(A) If some Ω occurs on the path P_Y , then we may adjoin either Ω_1 or Ω_2 as the sole successor of Y .

(B) If some Γ occurs on the path P_Y , then we may adjoin Γ_1 as the left successor of Y and Γ_2 as the right successor of Y .

A branch of a tableau for signed formulas is *closed* if it contains $T^\alpha(Y)$ and $\bar{T}^\beta(Y)$ for some $\alpha \geq \beta \in J$ and $Y \in F$. A tableau is called to be *closed* if every branch of the tableau is closed.

Definition 3.3. A branch P of a tableau is called *complete*, if for every Ω which occurs in P , both Ω_1 and Ω_2 occurs in P , and for every Γ which occurs in P , at least one Γ_1, Γ_2 occurs in P . We call a tableau T *completed* if every branch of T is either closed or complete.

§ 4. Consistency and completeness. We need the following definitions:

Definition 4.1. Let $v: F \rightarrow DL_2$ be a θ -valued valuation. A signed formula $T^\alpha(X)$ (resp. $\bar{T}^\alpha(X)$) is said to be *true under v* if $v(X)(\alpha) = 1$ (resp. $v(X)(\alpha) = 0$).

Definition 4.2. By an α -*proof* of a formula $X \in F$ is meant a closed tableau for $\bar{T}^\alpha(X)$.

Theorem 4.3. *The following statements hold:*

a) If $\bar{T}^\alpha(X)$ has a closed tableau, then X is an α -tautology.

b) If $T^\alpha(X)$ has a closed tableau, then X is not α -satisfiable.

Proof. a) Consider a tableau T and an interpretation v_0 whose domain includes at least all the propositional variables which occur in any point of T . Let us call a branch P of T true under v_0 if every term of P is true under v_0 , and we shall say that the tableau T is true under v_0 iff at least one branch of T is true under v_0 . If T' is a direct extension of T (i.e. T' can be obtained from T by one application of the operation (A) or (B)), then T' must be true in every interpretation in which T is true. From this it follows by mathematical induction that for any tableau T , if the origin is true under a given interpretation v_0 , then T must be true under v_0 . Now a closed tableau T cannot be true under any interpretation, because every branch contains $T^\alpha(Y)$, $\bar{T}^\beta(Y)$ for some $\alpha \geq \beta \in J$, $Y \in F$, hence $v(Y)(\alpha) = 1$, $v(X)(\alpha) = 0$ if this branch is true under a θ -valued valuation. From this it follows that if $\bar{T}^\alpha(X)$ has a closed tableau (i.e. $X \in F$ has an α -proof), then for every θ -valued valuation v we have $v(X)(\alpha) = 1$.

b) If $T^\alpha(X)$ has a closed tableau, for every α -valued valuation v , $v(X)(\alpha) = 0$, and so X is not α -satisfiable.

It therefore further follows that the tableau method is consistent in the sense that no $T^\alpha(X)$ and $\bar{T}^\alpha(X)$ are both α -provable (since no $T^\alpha(X)$ and $\bar{T}^\alpha(X)$ can both be α -tautologies).

Definition 4.4. We shall call *Hintikka set* a set S of formulas which satisfies the following conditions:

1) for every $\beta \leq \alpha \in J$ and $X \in V$ it is not the case that $T^\alpha(X) \in S$ and $\bar{T}^\beta(X) \in S$.

2) if $\Omega \in S$, then $\Omega_1 \in S$ and $\Omega_2 \in S$.

3) if $\Gamma \in S$, then $\Gamma_1 \in S$ or $\Gamma_2 \in S$.

Lemma 4.5. *Every Hintikka set is satisfiable, i.e. there exists a θ -valued valuation v such that if $T^\alpha(X) \in S$ then $v(X)(\alpha) = 1$ and if $\bar{T}^\alpha(X) \in S$ then $v(X)(\alpha) = 0$*

Proof. We remark that Lemma is equivalent to the statement that every Hintikka set can be extended to a saturated set.

Let $v_0: V \rightarrow DL_2$ be the interpretation given by

$$v_0(X)(\alpha) = \begin{cases} 1, & \text{if } T^\alpha(X) \in S, \\ 0, & \text{if } \bar{T}^\alpha(X) \in S, \\ 0 \text{ or } 1 & \text{if } T^\alpha(X) \notin S \text{ and } \bar{T}^\alpha(X) \notin S. \end{cases}$$

v_0 can be extended to a θ -valued valuation $v: F \rightarrow DL_2$. It is immediate that every signed variable which is in S , is true under v . Now consider an element X of S of degree greater than 0, and suppose all elements of S of lower degree than X are true under v . We wish to show that X must be true. Since X is of degree greater than zero, it must be either some Ω or some Γ ;

a) X is some Ω . Then Ω_1 and $\Omega_2 \in S$, Ω_1 and Ω_2 are of lower degree than X , hence by inductive hypothesis Ω_1 and Ω_2 are true under v . This implies that X must be true.

b) X is some Γ . Then Γ_1 or $\Gamma_2 \in S$, Γ_1 and Γ_2 are of lower degree than X , hence Γ_1 or Γ_2 must be true under v . Hence X must be true.

Lemma 4.6. *Any complete open (not closed) branch of any tableau is satisfiable.*

Proof. Let P be a complete open branch of a tableau T and S be the set of terms of P . Then the set S is by construction a Hintikka set, hence it is satisfiable by virtue of Lemma 4.5.

Theorem 4.7 (Completeness). *The following statements hold:*

- a) *If X is an α -tautology, then every completed tableau starting with $\overline{T}^\alpha(X)$ must be closed.*
- b) *Every α -tautology is provable by the tableau method.*

Proof. Let T be a complete tableau starting with $\overline{T}^\alpha(X)$. If T is not closed, there is a complete open branch P of T . By Lemma 4.6, the set of terms of P is satisfiable; then there is a θ -valued valuation $v: T \rightarrow DL_2$ which satisfies the terms of P . It follows that $\overline{T}^\alpha(X)$ is true under v , i.e. $v(X)(\alpha) = 0$, hence X cannot be an α -tautology. Hence if X is an α -tautology, then T must be closed.

Remark 1. If $S = \{S_1, S_2, \dots, S_n\}$ is a finite set of signed formulas, by a tableau for S is meant a tableau starting with

$$\begin{array}{c} S_1 \\ S_2 \\ \cdot \\ \cdot \\ S_n, \end{array}$$

and then extended using Rules A, B. Then S is unsatisfiable iff there exists a closed tableau for S .

Remark 2. Let us call a tableau atomically closed if every branch contains $\overline{T}^\alpha(X)$ and $T^\beta(X)$ for some $\alpha \leq \beta \in J$ and some propositional variable X . Then S is unsatisfiable iff there exists an atomically closed

tableau for S .

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Catedra de Matematici I
Institutul Politehnic Bucureşti
Spl. Independenţei 313
Bucharest
Romania