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Ikeda, Hiroshi

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Note on the 4-Manifolds in $\overline{C}(p, 2)$ with $p \geq 1$

By Hiroshi IKEDA^{*}

1. Introduction

K. Kobayashi and Y. Tsukui introduced the concept of the ball coverings of manifolds in [2]. Using it, the sets $C(p, q)$ and $\overline{C}(p, q)$ were defined for 4-manifolds in [1] and [3].

The purpose of this note is to clarify the construction of the elements of $\overline{C}(p, 2)$ with $p \geq 1$ in Theorem B. Roughly speaking, *for any element W of $\overline{C}(p, 2)$ with $p \geq 1$, there exists an element W' in $\overline{C}(p-1, 2)$ such that W is constructed from W' by removing a small open 4-ball in $\overset{\circ}{W}'$ and attaching a 1-handle in order to connect the two boundary components.*

A corresponding theorem to Theorem B is found in [1] and [3] for $\overline{C}(p, 1)$ with $p \geq 1$. We use the definitions and notations in [1], [2] and [3].

2. Preliminaries

Definition. Let W be an element of $C(p, q)$ and H a 3-ball properly embedded in W . H is called a *good handle* of W , if there exists a realization $W_F = (A, B; F)$ of W such that H is a connected component of F .

^{*} Department of Mathematics, Kobe University, Nada, Kobe, 657, Japan.

By the same argument as that of section 5 of [1] and [3], we obtain the following theorem immediately.

Theorem A. *Let W be an element of $\overline{C}(p, q)$. If W has a good handle there exists an element W' in $\overline{C}(p-1, q)$ such that*

(1) $W = (W' - \overset{\circ}{D}) \cup (D^3 \times I)$, where D is a 4-ball in $\overset{\circ}{W}'$, D^3 is a 3-ball and I is the unit interval $[0, 1]$,

(2) $(W' - \overset{\circ}{D}) \cap (D^3 \times I) = (W' - \overset{\circ}{D}) \cap (D^3 \times \dot{I})$,

(3) $D^3 \times 0 \subset \dot{W}'$ and $D^3 \times 1 \subset \dot{D}$.

3. Theorem B

Lemma. *Let W be an element of $C(p, q)$ and $W_F = (A, B; F)$ a realization of W . If $2p \geq q$, $\dot{F}$ contains a 2-sphere.*

Proof. Let n denote the number of the connected components of $\dot{F}$ which are not 2-spheres. We have $n \leq q$, since $H_1(\dot{F}) = 2q\mathbb{Z}$ by [1] and [3] and $\dot{F}$ is an orientable closed 2-manifold. On the other hand, the condition $\#(\dot{A} - \dot{F}) = 2(p+1)$ follows from $\#(\dot{A} - F) = \#F = p+1$ in [1] and [3]. Thus, we have $\#\dot{F} = 2p+1$. By our assumption $2p \geq q$, we obtain $\#\dot{F} \geq q+1$. Therefore, $\#\dot{F}$ is really greater than n , that is, at least one connected component of $\dot{F}$ should be a 2-sphere.

Theorem B. *Let W be an element of $\overline{C}(p, 2)$. If $p \geq 1$, W has a good handle.*

Proof. Let $W_F = (A, B; F)$ be a realization of W . By Lemma, we can find a 2-sphere S in $\dot{F}$.

Case 1. Suppose S bounds a 3-ball H in $\dot{A} - (\dot{F} - S)$. Then, H is a

connected component of F or F' , where $F' = \dot{A} - \overset{\circ}{F}$. If H is a connected component of F , W_F is a required realization of W for H to be a good handle of W . If H is contained in F' , take the induced realization $W_{F'} = (A, C; F')$ of W (for induced realizations, see [3]). Then, $W_{F'}$ is automatically a required realization of W .

Case 2. Suppose S does not bound a 3-ball in $\dot{A} - (\dot{F} - S)$ for any 2-sphere S in $\dot{F}$. Let $X_1, \dots, X_{2p+2}$ denote the closures of the connected components of $\dot{A} - \dot{F}$. Since $\dot{A}$ is a 3-sphere, we can assume that $\dot{X}_1$ and $\dot{X}_2$ are connected and $X_1 \neq X_2$.

Step 1. In this step, we show that $X_1 \cap X_2 = \emptyset$ and each of $\dot{X}_1$ and $\dot{X}_2$ is a torus of genus 1. From the construction of X_i 's, $X_1 \cap X_2 = \emptyset$ follows from $\dot{X}_1 \cap \dot{X}_2 = \emptyset$. Thus, we prove $\dot{X}_1 \cap \dot{X}_2 = \emptyset$. Since $\dot{F}$ is a 2-manifold, $\dot{X}_1 = \dot{X}_2$ provided $\dot{X}_1 \cap \dot{X}_2 \neq \emptyset$. Then, we have $\dot{A} = X_1 \cup X_2$. This contradicts to our hypothesis $p \geq 1$. Now, we prove the second statement. Since $\dot{X}_1$ and $\dot{X}_2$ are connected, they are not 2-spheres. $q = 2$ implies $H_1(\dot{F}) = 4\mathbb{Z}$. Thus, $H_1(\dot{X}_1) + H_1(\dot{X}_2) = 4\mathbb{Z}$. Therefore, each of $\dot{X}_1$ and $\dot{X}_2$ should be a torus of genus 1, because $\dot{X}_1 \cap \dot{X}_2 = \emptyset$.

Step 2. Suppose X_3 and X_4 satisfy the conditions $X_1 \cap X_3 = \dot{X}_1 \cap \dot{X}_3 \neq \emptyset$ and $X_2 \cap X_4 = \dot{X}_2 \cap \dot{X}_4 \neq \emptyset$. We show $X_3 \neq X_4$. Suppose $X_3 = X_4$. If $\dot{X}_3$ does not contain a 2-sphere, we have $\dot{X}_3 = \dot{X}_1 \cup \dot{X}_2$. Then, $\dot{A} = X_1 \cup X_2 \cup X_3$. This is a contradiction to $2p + 2 \geq 4$. Thus, there exists a 2-sphere S in $\dot{X}_3$. Let D be the 3-ball bounded by S in $\dot{A} - (X_1 \cup X_2 \cup X_3)^\circ$. Since the connected components of $D \cap \dot{F}$ are 2-spheres, at least one of them, say S' , should bound a 3-ball in $D - (\dot{F} - S')$. This contradicts to the assumption of Case 2. Therefore, $X_3 \neq X_4$.

Further, it is not hard to see that $\dot{X}_i$ does not contain a torus and

$\# \dot{X}_i \geq 2$ for $i = 5, \dots, 2p + 2$.

By the same argument as above, we obtain the same conditions for $\dot{B} - \dot{F}$. That is, let $Y_1, \dots, Y_{2p+2}$ be the closures of the connected components of $\dot{B} - \dot{F}$; then,

- (1) $Y_1 \cap Y_2 = \emptyset$ and each of $\dot{Y}_1$ and $\dot{Y}_2$ is a torus of genus 1,
- (2) $Y_1 \cap Y_3 = \dot{Y}_1 \cap \dot{Y}_3 \neq \emptyset \neq \dot{Y}_2 \cap \dot{Y}_4 = Y_2 \cap Y_4$, $Y_3 \neq Y_4$ and each of $\dot{Y}_3$ and $\dot{Y}_4$ contains a 2-sphere.
- (3) $\dot{Y}_i$ does not contain a torus and $\# \dot{Y}_i \geq 2$ for $i = 5, \dots, 2p + 2$.

Step 3. Without loss of generality, we can assume $\dot{X}_1 = \dot{Y}_1$. First, we claim that $X_1 \subset F$ if and only if $Y_1 \subset F$. $X_1 \subset F$ means that X_1 is identified with Y_1 or Y_3 . But, it is impossible to identify X_1 and Y_3 , for $\dot{X}_1$ is not homeomorphic to $\dot{Y}_3$. Thus, $Y_1 \subset F$. Suppose $X_1 \not\subset F$. Then, we have $\dot{W} = X_1 \cup Y_1$, because $X_1 \cup Y_1$ is a closed 3-manifold in $\dot{W}$ and $\dot{W}$ is connected. This contradicts to $\#(\dot{A} - F) = p + 1 \geq 2$. Therefore, it remains to deal with the case $X_1 \cup X_2 \subset F$. Now, we have to construct a 3-sphere from a subset of $\{X_3, \dots, X_{2p+2}\} \cup \{Y_3, \dots, Y_{2p+2}\}$ by identifying only the boundaries of the elements. This is, clearly, impossible, because $\# \dot{X}_i \geq 2$ and $\# \dot{Y}_i \geq 2$ for $i = 3, \dots, 2p + 2$. This completes the proof of Theorem B.

References

- [1] H. Ikeda and M. Yamashita: Closed 4-manifolds covered by three 4-balls, Yokohama Math. Jour., 26 (1978), 109-117.
- [2] K. Kobayashi and Y. Tsukui: The ball coverings of manifolds, Jour. Math. Soc. Japan, 28 (1976), 133-143.
- [3] M. Yamashita and H. Ikeda: 4-manifolds of covering number 2, Math. Seminar Notes, 4 (1976), 105-111.