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Fixed Point Theorems for a Contractive Type Mapping

Dedicated to Professor Kiyoshi Iséki on his 60th birthday

By B. E. RHOADES

Two fixed point theorems are established for functions satisfying a general contractive type condition. These theorems generalize the corresponding results of Ćirić, Guseman, Iséki, Khazanchi, and Sehgal.

Let N denote the set of all positive integers. In this paper we shall concern with a self mapping f of a complete metric space (X, d) satisfying the following contractive type condition:

(1) There exist nonnegative real valued functions α, β, γ, δ, ε on $X \times X$ and a mapping $n:X \to N$ such that

$$\sup_{x,\, y \in X} [2(\alpha(x, y) + \beta(x, y) + \delta(x, y) + \varepsilon(x, y)) + \gamma(x, y)] = \lambda < 1$$

and

$$d(f^{n(x)}(x), f^{n(x)}(y)) \leq \alpha(x, y)d(x, f^{n(x)}(y)) + \beta(x, y)d(y, f^{n(x)}(x)) + \gamma(x, y)d(x, y) + \delta(x, y)d(x, f^{n(x)}(x)) + \varepsilon(x, y)d(y, f^{n(x)}(x))$$

for all x, y in X.

First it shall be noted that (1) has two other equivalent forms:

(2) There exist a number λ, $0 \leq \lambda < 1$, and a mapping $n:X \to N$ such that

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$$d(f^{n(x)}(x),\ f^{n(x)}(y)) \le \lambda \max\{d(x,\ y),\ [d(x,\ f^{n(x)}(y)) + d(y,\ f^{n(x)}(x))]/2,\ [d(x,\ f^{n(x)}(x)) + d(y,\ f^{n(x)}(y))]/2\}$$

for all $x,\ y$ in X.

(3) There exist a number $\lambda,\ 0 \le \lambda < 1$, and a mapping $n:X \to N$ such that

$$d(f^{n(x)}(x),\ f^{n(x)}(y)) \le (\lambda/2) \max\{2d(x,\ y),\ d(x,\ f^{n(x)}(y)),\ d(y,\ f^{n(x)}(x)),\ d(x,\ f^{n(x)}(x)),\ d(y,\ f^{n(x)}(y))\}$$

for all $x,\ y$ in X.

One may write

$$\begin{aligned} &\alpha d(x,\ f^{n(x)}(y)) + \beta d(y,\ f^{n(x)}(x)) \\ &= 2\alpha d(x,\ f^{n(x)}(y))/2 + 2\beta d(y,\ f^{n(x)}(x))/2 \\ &\le (2\alpha + 2\beta) \max\{d(x,\ f^{n(x)}(y))/2,\ d(y,\ f^{n(x)}(x))/2\}. \end{aligned}$$

Using a similar argument for $\delta d(x,\ f^{n(x)}(x)) + \varepsilon d(y,\ f^{n(x)}(y))$ it easily follows that (1) implies (3). Clearly (3) implies (2). To show that (2) implies (1), we set

$$M(x,\ y) = \max\{d(x,\ y),\ [d(x,\ f^{n(x)}(y)) + d(y,\ f^{n(x)}(x))]/2,\ [d(x,\ f^{n(x)}(x)) + d(y,\ f^{n(x)}(y))]/2\},$$

and define $A_1 = \{(x,\ y) \in X \times X \mid M(x,\ y) = d(x,\ y)\}$, $A_2 = \{(x,\ y) \in X \times X \mid M(x,\ y) = [d(x,\ f^{n(x)}(y)) + d(y,\ f^{n(x)}(x))]/2\}$, and $A_3 = \{(x,\ y) \in X \times X \mid M(x,\ y) = [d(x,\ f^{n(x)}(x)) + d(y,\ f^{n(x)}(y))]/2\}$. Then it is possible to choose a partition $\{S_1,\ S_2,\ S_3\}$ of X such that $S_i \subset A_i$, $i = 1,\ 2,\ 3$. Hence letting

$$\alpha(x,\ y) = \beta(x,\ y) = \begin{cases} \lambda/2 & \text{if } (x,\ y) \in S_2 \\ 0 & \text{otherwise,} \end{cases}$$

$$\gamma(x, y) = \begin{cases} \lambda & \text{if } (x, y) \in S_1 \\ 0 & \text{otherwise,} \end{cases} \qquad \delta(x, y) = \varepsilon(x, y) = \begin{cases} \lambda/2 & \text{if } (x, y) \in S_3 \\ 0 & \text{otherwise,} \end{cases}$$

we obtain (1). Thus (1), (2), and (3) are equivalent to each other.

As a consequence of Lemma of [1], we have the following:

Lemma 1. *If* $f: X \to X$ *satisfies* (1), *then, for each* $x \in X$, $r(x) = \sup_n d(f^n(x), x)$ *is finite.*

We need the following lemma.

Lemma 2. *Suppose that* $f: X \to X$ *satisfies* (3). *If* $\{x_m\}$ *is a sequence in* X *such that* $x_{n+1} = f^{n(x_m)}(x_m)$ *for all* $m \in N \cup \{0\}$, *then*

$$\sup\{d(x_m, f^k(x_m)) \mid k \in N\} \le \lambda^m r(x_0). \tag{4}$$

for all $m \in N \cup \{0\}$.

Proof. Clearly (4) is true for $m = 0$. Assume (4) holds for an $m \in N \cup \{0\}$. Then for each $k \in N$,

$$\begin{aligned} d(x_{m+1}, f^k(x_{m+1})) &= d(f^{n(x_m)}(x_m), f^{n(x_m)}f^k(x_m)) \\ &\le \frac{\lambda}{2}\max\{d(x_m, f^k(x_{m+1})), d(f^k(x_m), x_{m+1}), d(x_m, x_{m+1}), \\ &\qquad d(f^k(x_m), f^k(x_{m+1})), 2d(x_m, f^k(x_m))\} \\ &\le \frac{\lambda}{2}\max\{[d(x_m, x_{m+1}) + d(x_{m+1}, f^k(x_{m+1}))], \\ &\qquad [d(f^k(x_m), x_m) + d(x_m, x_{m+1})], d(x_m, x_{m+1}), \\ &\qquad [d(f^k(x_m), x_m) + d(x_m, f^k(x_{m+1}))], 2d(x_m, f^k(x_m))\} \\ &= \frac{\lambda}{2}\max\{[d(x_m, f^{n(x_m)}(x_m)] + d(x_{m+1}, f^k(x_{m+1}))], \\ &\qquad [d(f^k(x_m), x_m) + d(x_m, f^{n(x_m)}(x_m))], \\ &\qquad [d(f^k(x_m), x_m) + d(x_m, f^{k+n(x_m)}(x_m))], 2d(x_m, f^k(x_m))\} \end{aligned}$$

$$\leq \frac{\lambda}{2}\max\{[\lambda^m r(x_0) + d(x_{m+1}, f^k(x_{m+1}))], 2\lambda^m r(x_0)\},$$

which implies $d(x_{m+1}, f^k(x_{m+1})) \leq \lambda^{m+1} r(x_0)$.

Theorem 1. *If $f: X \to X$ satisfies (1), then f has a unique fixed point z and $\lim_n f^n(x_0) = z$ for each $x_0 \in X$.*

Proof. Define $\{x_n\}$ by $x_{m+1} = f^{n(x_m)}(x_m)$ for all m in $N \cup \{0\}$. Since $\lambda < 1$, from Lemma 2, it follows that $\{x_n\}$ is Cauchy, hence convergent. Call the limit z. Suppose $\varepsilon = d(z, f^{n(z)}(z)) > 0$. Then we can find an $m \in N$ such that $d(z, x_n) < (1 - \lambda)\varepsilon/2$ and $\lambda^n r(x_0) < (1 - \lambda)\varepsilon/2$ for all $n \geq m$. Therefore using (3) and (4) we have

$$\begin{aligned}
\varepsilon &= d(z, f^{n(z)}(z)) \\
&\leq d(z, x_m) + d(x_m, f^{n(z)}(x_m)) + d(f^{n(z)}(x_m), f^{n(z)}(z)) \\
&< \frac{(1 - \lambda)\varepsilon}{2} + \frac{(1 - \lambda)\varepsilon}{2} + d(f^{n(z)}(x_m), f^{n(z)}(z)) \\
&\leq (1 - \lambda)\varepsilon + \frac{\lambda}{2}\max\{d(z, f^{n(z)}(x_m)), d(x_m, f^{n(z)}(z)), \\
&\qquad d(z, f^{n(z)}(z)), d(x_m, f^{n(z)}(x_m)), 2d(z, x_m)\} \\
&\leq (1 - \lambda)\varepsilon + \frac{\lambda}{2}\max\{d(z, x_m) + d(x_m, f^{n(z)}(x_m)), \\
&\qquad d(x_m, z) + d(z, f^{n(z)}(z)), 2d(z, x_m)\}.
\end{aligned}$$

Since $d(z, x_m) < \varepsilon/2$ and $d(x_m, f^{n(z)}(x_m)) \leq \lambda^m r(x_0) < \varepsilon/2$, this implies

$$\begin{aligned}
\varepsilon &< (1 - \lambda)\varepsilon + \frac{\lambda}{2}\max\{\frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \frac{\varepsilon}{2} + \varepsilon, \varepsilon\} \\
&< \varepsilon,
\end{aligned}$$

a contradiction. Thus $f^{n(z)}(z) = z$.

Suppose z and w are distinct fixed points of $f^{n(z)}$. Then, from (1),

$$\begin{aligned}
d(z, w) &= d(f^{n(z)}(z), f^{n(z)}(w)) \\
&\leq [\alpha(z, w) + \beta(z, w) + \gamma(z, w)]d(z, w) < d(z, w),
\end{aligned}$$

a contradiction. Therefore z is unique. $z = f^{n(z)}(z)$ implies $f(z) =$

$f^{n(z)}(f(z))$ so that $f(z)$ is also a fixed point of $f^{n(z)}$. Hence $z = f(z)$. It remains to show that $\lim_n f^n(x_0) = z$.

Let $\rho(x) = \max\{d(x, f^k(x)) \mid k = 1, 2, \cdots, n(x)\}$ and set $n = n(z)$. For each integer $m > n$, write $m = qn + s$, where $q \geq 1$ and $0 \leq s < n$. Assume $f^m(x_0) \neq z$ for each m. From (1), with $\alpha - \varepsilon$ evaluated at $(f^{(q-1)n+s}(x_0), z)$,

$$\begin{aligned} d(f^m(x_0), z) &= d(f^{qn+s}(x_0), f^n(z)) \\ &\leq \alpha d(f^{(q-1)n+3}(x_0), f^n(z)) + \beta d(z, f^m(x_0)) \\ &\quad + \gamma d(f^{(q-1)n+s}(x_0), z) + \delta d(f^{(q-1)n+s}(x_0), f^m(x_0)), \end{aligned}$$

so that $d(f^m(x_0), z) \leq \lambda d(f^{(q-1)n+s}(x_0), z)$, and hence $d(f^m(x_0), z) \leq \lambda^q(x_0)$, which tends to zero as $q \to \infty$.

An analysis of the proof of Theorem 1 shows that one needs only to have f and its iterates belong to some subset B of X and f satisfy (1) on B. We state this observation formally.

Theorem 2. *Let f be a selfmapping of a complete metric space (X, d). Suppose there exists a subset $B \subset X$ such that*

(i) $f(B) \subset B$,

(ii) *f satisfies* (1) *over B,*

(iii) *for some $x_0 \in B$, $\mathrm{cl}\{f^n(x_0) : n \geq 1\} \subset B$, where* cl *denotes the closure.*

Then there is a unique $z \in B$ satisfying $f(z) = z$ and $f^n(y_0) \to z$ for each $y_0 \in B$. Furthermore, if f satisfies (1) *over X, then z is unique in X and $f^n(x_0) \to z$ for each $x_0 \in X$.*

Remark 1. The contractive definition (2) of this paper is contractive

definition (94) of [5].

Remark 2. The special case of Theorem 1 with $\alpha = \beta = \delta = \varepsilon = 0$, γ a constant, and the added assumption that f be continuous, appears in [6].

Remark 3. The special case of Theorem 2 with $\alpha = \beta = \delta = \varepsilon = 0$ appears in [2].

Remark 4. The special cases of Theorems 1 and 2 for $\delta = \varepsilon = 0$, α, β, γ constants, appears in [3].

Remark 5. It is easy to see that Theorems 1 and 2 remain valid if (X, d) is f-orbitally complete. Hence Theorem 5 of [1] is then a special case of Theorem 2 with B the closure of the orbit of a point x_0.

Remark 6. To show that (1) is more general than the definitions appearing in [3] and [4] observe that Iséki's definition is included in the following:

(5) There exist a real number λ, $0 \leq \lambda < 1$, and a mapping $n : X \to N$ such that for all $x, y \in X$,

$$d(f^{n(x)}(x), f^{n(x)}(y)) \leq \lambda \max \{2d(x, y), [d(x, f^{n(x)}(x)) + d(y, f^{n(x)}(y))]/2, [d(x, f^{n(x)}(y)) + d(y, f^{n(x)}(x))]/2\},$$

and that Khazanchi's definition is included in the following:

(6) There exist a real number λ, $0 \leq \lambda < 1$, and a mapping $n : X \to N$ such that, for all $x, y \in X$,

$$d(f^{n(x)}(x), f^{n(x)}(y)) \leq h \max \{d(x, y), [d(x, f^{n(x)}(y)) + d(y, f^{n(x)}(x))]/2\}.$$

Both (5) and (6) are special cases of (2).

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