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Mappings with a Common Fixed Point

Dedicated to Professor Kiyoshi Iséki on his 60th birthday

By Brian FISHER

In a paper by Jungck, see [1], he proves the following theorem:

Theorem 1. *Let T be a continuous mapping of a complete metric space (X, ρ) into itself. Then T has a fixed point in X if and only if there exists a mapping A of X into the range TX of T, which commutes with T and satisfies the inequality*

$$\rho(Ax, Ay) \leq \alpha\rho(Tx, Ty)$$

for all x, y in X, where $0 < \alpha < 1$. Indeed, T and A then have a unique fixed point.

We will now prove the following generalization:

Theorem 2. *Let S and T be continuous mappings of a compelte metric space (X, ρ) into itself. Then S and T have a common fixed point in X if and only if there exists a continuous mapping A of X into $SX \cap TS$, which commutes with S and T and satisfies the inequality*

$$\rho(Ax, Ay) \leq \alpha\rho(Sx, Ty)$$

for all x, y in X, where $0 < \alpha < 1$. Indeed, S, T and A then have a

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unique common fixed point.

Proof. We will first of all prove the necessity of the condition. Suppose $Sz = z = Tz$ for some z in X. Define a mapping A of X into X by $Ax = z$, for all x in X. A is obviously a continuous mapping of X into $SX \cap TX$ and

$$ASx = SAx = ATx = TAx = z$$

for all x in X. Thus A commutes with S and T. Further, for any α, with $0 < \alpha < 1$

$$\rho(Ax, Ay) = 0 \leq \alpha\rho(Sx, Ty)$$

for all x, y in X. This proves the necessity of the condition.

We will now prove the sufficiency of the condition. Suppose such a mapping A exists. Let x_0 be an arbitrary point in X. Since AX is contained in SX we can choose a point x_1 in X such that $Sx_1 = Ax_0$. Since AX is also contained in TX we can now choose a point x_2 in X such that $Tx_2 = Ax_1$. In general, having chosen the point x_{2n-2} we choose a point x_{2n-1} such that $Sx_{2n-1} = Ax_{2n-2}$ and then choose a point x_{2n} such that $Tx_{2n} = Ax_{2n-1}$. We then have

$$\begin{aligned}
\rho(Ax_{2n}, Ax_{2n-1}) &= \rho(Ax_{2n-1}, Ax_{2n}) \\
&\leq \alpha\rho(Sx_{2n-1}, Tx_{2n}) \\
&= \alpha\rho(Ax_{2n-2}, Ax_{2n-1}) \\
&= \alpha\rho(Ax_{2n-1}, Ax_{2n-2}),
\end{aligned}$$

$$\begin{aligned}
\rho(Ax_{2n+1}, Ax_{2n}) &\leq \alpha\rho(Sx_{2n+1}, Tx_{2n}) \\
&= \alpha\rho(Ax_{2n}, Ax_{2n-1})
\end{aligned}$$

for $n = 1, 2, \cdots$. Hence $\rho(Ax_{n+1}, A_n) \leq \alpha\rho(Ax_n, Ax_{n-1}) \leq \cdots \leq \alpha^n\rho(Ax_1, Ax_0)$ for all $n = 1, 2, \cdots$. It follows that $\{Ax_n\}$ is a Cauchy sequence, and so it has a limit z in X. Since the sequences $\{Sx_{2n+1}\}$

and $\{Tx_{2n}\}$ are subsequences of $\{Ax_n\}$, they have the same limit z. Since A and S are continuous, we have $Sz = \lim_{n\to\infty} SAx_{2n+1} = \lim_{n\to\infty} ASx_{2n+1} = Az$.

Similarly we have

$$Az = Tz.$$

It now follows that

$$\begin{aligned}\rho(Az, A^2z) &\le \alpha\rho(Sz, TAz)\\ &= \alpha\rho(Sz, ATz)\\ &= \alpha\rho(Az, A^2z)\end{aligned}$$

and since $\alpha < 1$ we must have

$$Az = A^2z.$$

Finally, putting $Az = z_1$, we have

$$Az_1 = A^2z = z_1 = ASz = SAz = ATz = TAz = Tz_1$$

and so z_1 is a fixed point of S, T and A. This proves the sufficiency of the condition.

To show that z_1 is a unique fixed point of S, T and A let us suppose that z_2 is also a common fixed point. Then

$$\begin{aligned}\rho(z_1, z_2) &= (Az_1, Az_2)\\ &\le \alpha\rho(Sz_1, Tz_2)\\ &= \alpha\rho(z_1, z_2).\end{aligned}$$

Since $\alpha < 1$, it follows that $z_1 = z_2$ and so the common fixed point is unique. This completes the proof of the theorem.

Reference

[1] Gerald Jungck: Commuting mappings and fixed points, Amer. Math. Monthly, 83 (1976), 261-263.

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