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Generalizations of Hegedüs' Fixed Point Theorem

Dedicated to Professor Kiyoshi Iséki on his 60th birthday

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Recently Hegedüs has obtained in [1] the following interesting generalization of the Banach contraction theorem, in which ω denotes the set of all nonnegative integers.

Let f *be a mapping of a nonempty complete metric space* (X, d) *into itself. If the diameter* $\delta(O_f(x))$ *of the orbit* $O_f(x) = \{f^n(x) \mid x \in \omega\}$ *is finite for each* $x \in X$, *and if there exists an* α *such that* $0 \leq \alpha < 1$ *and*

$$d(f(x), f(y)) \leq \alpha \cdot \delta(O_f(x) \cup O_f(y))$$

for all $x, y \in X$, *then* f *has a unique fixed point* $a \in X$ *and the sequence* $\{f^n(x)\}_{n \in \omega}$ *converges to* a *for each* $x \in X$.

The purpose of this note is to state two generalizations of this theorem.

Let f be a mapping of a metric space (X, d) into itself. A point x of X is said to be *regular for* f if $\delta(O_f(x))$ is finite.

One of our main results is the following:

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Theorem 1. *Let f be a mapping of a metric space (X, d) into itself, and φ a nondecreasing nonnegative real valued function on $\mathbb{R}_+$, the set of all nonnegative real numbers, which is upper semicontinuous from the right and satisfies the inequality $\varphi(t) < t$ for every $t > 0$. Suppose there exists a regular point $x_0 \in X$ for f such that some subsequence of the sequence $\{f^n(x_0)\}_{n\in\omega}$ converges to a regular point $a \in X$ for which the following inequality holds*

$$(*) \qquad d(f(x), f(y)) \le \varphi(\delta(O_f(x) \cup O_f(y)))$$

for all $x, y \in O_f(x_0) \cup O_f(a)$. Then a is a fixed point of f and the sequence $\{f^n(x_0)\}_{n\in\omega}$ converges to a. Moreover if $(*)$ holds for all regular points $x, y \in X$ for f, then a is a unique fixed point of f.

Proof. Since x_0 is regular for f, $\delta_n = \delta(O_f(f^n(x_0)))$ is finite for each $n \in \omega$. Clearly the sequence $\{\delta_n\}_{n\in\omega}$ is nonincreasing, and hence it converges to some $\delta \ge 0$. Suppose $\delta > 0$. Then $\varphi(\delta) < \delta$, and so $\varphi(\delta_n) < \delta$ for some $n \in \omega$ since φ is upper semicontinuous from the right. However we have, for all $i, j \ge n + 1$,

$$\begin{aligned} d(f^i(x_0), f^j(x_0)) &= d(f(f^{i-1}(x_0)), f(f^{j-1}(x_0))) \\ &\le \varphi(\delta(O_f(f^{i-1}(x_0)) \cup O_f(f^{j-1}(x_0))) \\ &\le \varphi(\delta(O_f(f^n(x_0))), \end{aligned}$$

which implies $\delta_{n+1} \le \varphi(\delta_n) < \delta$, a contradiction. So we have $\delta = 0$ which shows that $\{f^n(x_0)\}_{n\in\omega}$ is a Cauchy sequence. Therefore $\{f^n(x_0)\}_{n\in\omega}$ converges to a. Suppose now $t = \delta(O_f(a)) > 0$. Then, sicne $\varphi(t) < t$, we can find an α with $\varphi(t)/t < \alpha < 1$. Since φ is upper semicontinuous from the right, there exists a $\gamma > 0$ such that $\varphi(t + \varepsilon)/(t + \varepsilon) < \alpha$ for all ε with $0 < \varepsilon < \gamma$. Choose an ε with $0 < \varepsilon < \min\{(1 - \alpha)t/(1 + \alpha), \gamma\}$. We can find then an $n \in \omega$ such that $d(a, f^m(x_0)) < \varepsilon$ for all $m \ge n$. So if $m \geq 1$, we have

$$(**)\qquad \begin{aligned} d(a, f^m(a)) &\leq d(a, f^{n+1}(x_0)) + d(f(f^n(x_0)), f^m(a)) \\ &< \varepsilon + \varphi(\delta(O_f(f^n(x_0)) \cup O_f(f^{m-1}(a))) \\ &\leq \varepsilon + \varphi(\max\{2\varepsilon, \delta(O_f(a)) + \varepsilon\}) \\ &= \varepsilon + \varphi(t + \varepsilon) \\ &< \varepsilon + \alpha(t + \varepsilon). \end{aligned}$$

On the other hand, it follows from

$$\delta(O_f(a)) = \max\{\sup\{d(a, f^m(a)) \mid m \in \omega\}, \delta(O_f(f(a)))\}$$

and $\delta(O_f(f(a))) \leq \delta(O_f(a))$, that

$$t = \delta(O_f(a)) = \sup\{d(a, f^m(a)) \mid m \in \omega\}.$$

Hence from (**), we obtain $t \leq \varepsilon + \alpha(t + \varepsilon)$ which yields a contradiction

$$t \leq \frac{1 + \alpha}{1 - \alpha}\varepsilon < t.$$

Thus we have $t = 0$ or equivalently $f(a) = a$. Suppose now that (*) holds for all regular points $x, y \in X$ for f. If $x, y \in X$ are distinct fixed points of f, then they are regular for f, and so we have

$$d(x, y) = d(f(x), f(y)) \leq \varphi(\delta(O_f(x) \cup O_f(y))) = \varphi(d(x, y)) < d(x, y),$$

a contradiction. This completes the proof.

We shall now state another main result which is a consequence of Theorem 1.

Theorem 2. *Let f be a mapping of a nonempty complete metric space (X, d) into itself and φ a nondecreasing nonnegative real valued function on $\mathbb{R}_+$ which is upper semicontinuous from the right and satisfies the inequality $\varphi(t) < t$ for every $t > 0$. If every point of X is regular for f and (*) holds for all $x, y \in X$, then f has a unique fixed point* *and the sequence* $\{f^n(x)\}_{n\in\omega}$ *converges to the fixed point for each* $x \in X$.

Proof. Let x_0 be an arbitrary point of X. Then, since x_0 is regular for f, $\{f^n(x_0)\}_{n\in\omega}$ is a Cauchy sequence as is shown in the proof of Theorem 1. Therefore $\{f^n(x_0)\}_{n\in\omega}$ converges to some $a \in X$. Thus the conclusion follows from Theorem 1.

Let α be in $[0, 1)$. Then it is obvious that the function φ on $\mathbb{R}_+$ defined by $\varphi(t) = \alpha t$ for all $t \in \mathbb{R}_+$ is nondecreasing, upper semicontinuous from the right and satisfies the inequality $\varphi(t) < t$ for every $t > 0$. Thus Hegedüs' theorem is a consequence of Theorem 2. The following example shows that Hegedüs' theorem is in fact a special case of Theorem 2.

Example 1. Let $X = [0, 1]$ and let $d(x, y) = |x - y|$ for all x, y in X. Consider the functions f and φ defined as follows:

$$f(x) = \begin{cases} \left(1 - \dfrac{n+2}{2^{n+2}}\right)x + \dfrac{1}{2^{n+2}} & \text{if } \dfrac{1}{n+1} \le x \le \dfrac{1}{n} \quad (n \ge 1), \\ 0 & \text{if } x = 0, \end{cases}$$

$$\varphi(x) = \begin{cases} f(x) & \text{if } x \in X, \\ \dfrac{3}{4} & \text{if } x \in \mathbb{R}_+ \setminus X. \end{cases}$$

Obviously φ is a continuous nondecreasing function satisfying the inequality $\varphi(t) < t$ for every $t > 0$. Since $\delta(O_f(x)) = x$ for each $x \in X$, if $y \le x$ then

$$d(f(x), f(y)) \le f(x) = \varphi(x) = \varphi(\delta(O_f(x) \cup O_f(y))).$$

Thus Theorem 2 can be applied. However f is not a generalized Banach contraction in the sense of Hegedüs [1]. In fact, for each $\alpha \in [0, 1)$ we can find an $n \ge 1$ such that

$$\alpha < 1 - \frac{n+2}{2^{n+2}},$$

and hence we have

$$\alpha\cdot\delta(O_f(\tfrac{1}{n}) \cup O_f(0)) = \frac{\alpha}{n} < \left(1 - \frac{n+2}{2^{n+2}}\right)\frac{1}{n} < f(\tfrac{1}{n})$$
$$= |f(\tfrac{1}{n}) - f(0)| = d(f(\tfrac{1}{n}), f(0)).$$

As the following example shows, Theorem 1 essentially extends Theorem 2.

Example 2. Let X denote the set of all rationals in $[0, 1]$, and let $d(x, y) = |x - y|$ for all x, y in X. Then the mapping f in Example 1 defines a mapping of X into itself which will be denoted by the same symbol f. Since the sequence $\{f^n(0)\}_{n\in\omega}$ converges to 0, Theorem 1 can be applied, but not Theorem 2.

Reference

[1] M. Hegedűs: A new generalization of Banach's contraction principle, to appear in Acta Sci. Math., Szeged.