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(Citation)
Mathematics Seminar Notes,7(1):123-131

(Issue Date)
1979

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/E0001461

(URL)
https://hdl.handle.net/20.500.14094/E0001461

MATHEMATICS SEMINAR NOTES
Vol. 7 (1979)

On Some Recent Results on Fixed Points II

Dedicated to Professor Kiyoshi Iséki on his 60th birthday

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The purpose of this note is to prove two common fixed point theorems in L-spaces which extend recent results obtained by Fisher [1], Park [4], and Singh [5].

For terminologies and notations not defined here the reader may consult [2], [3] and [5]. By an *o-metric* on a set X, we mean a nonnegative real valued function d on $X \times X$ with the property that $d(x, y) = 0$ if and only if $x = y$.

Theorem 1. *Let* $(X, \to)$ *be an* L-*space*, d *a continuous o-metric on* X, *and* P, Q, S, T *be mappings of* X *into itself such that* $PS = SP$ *and* $QT = TQ$. *Suppose there exists an upper semicontinuous function* $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ *such that*

$$\max\{\varphi(t, t, t, 0, 0), \varphi(0, 0, t, t, 0), \varphi(0, t, 0, 0, t)\} < t$$

for all $t > 0$ *and such that*

$$d(Px, Qy) \leq \varphi(d(Sx, Ty), d(Sx, Qy), d(Px, Ty), d(Px, Sx), d(Ty, Qy))$$

for all x, y *in* X. *If there exist sequences* $\{x_n\}_{n\in\omega}$ *and* $\{y_n\}_{n\in\omega}$ *in*

Received February 20, 1979.

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X and points a, b in X such that S is continuous at a, T is continuous at b, and such that $Px_n \to a$, $Sx_n \to a$, $Qy_n \to b$, and $Ty_n \to b$, then Pa ($= Qb = Sa = Tb$) is a unique common fixed point of P and S, and at the same time, it is a unique common fixed point of Q and T (hence Pa is a unique common fixed point of P, Q, S, and T).

Proof. The continuity of S at a ensures the existence of a subsequence $\{n_i\}_{i\in\omega}$ of $\{n\}_{n\in\omega}$ with $SSx_{n_i} \to Sa$. Since $Px_{n_i} \to a$, there is a subsequence s of $\{x_{n_i}\}_{i\in\omega}$ such that $SP\mathrm{s} \to Sa$. Clearly we may assume without loss of generality that $SSx_{n_i} \to Sa$ and $SPx_{n_i} \to Sa$. Since $Ty_{n_i} \to b$ and $Qy_{n_i} \to b$, the same argument shows that $TT\mathrm{t} \to Tb$ and $TQ\mathrm{t} \to Tb$ for some subsequence t of $\{y_{n_i}\}_{i\in\omega}$. Thus we may assume that $SSx_{n_i} \to Sa$, $PSx_{n_i} \to Sa$, $TTy_{n_i} \to Tb$, and $QTy_{n_i} \to Tb$. Since d is continuous, by choosing subsequences repeatedly, we can conclude that

$$d(PSx_{n_i}, QTy_{n_i}) \to d(Sa, Tb), \quad d(SSx_{n_i}, TTy_{n_i}) \to d(Sa, Tb),$$
$$d(SSx_{n_i}, QTy_{n_i}) \to d(Sa, Tb), \quad d(PSx_{n_i}, TTy_{n_i}) \to d(Sa, Tb),$$
$$d(PSx_{n_i}, SSx_{n_i}) \to d(Sa, Sa) = 0, \text{ and } d(TTy_{n_i}, QTy_{n_i}) \to d(Tb, Tb) = 0$$

for some subsequence $\{n_i\}_{i\in\omega}$ of $\{n\}_{n\in\omega}$. On the other hand, we have

$$d(PSx_{n_i}, QTy_{n_i}) \le \varphi(d(SSx_{n_i}, TTy_{n_i}), d(SSx_{n_i}, QTy_{n_i}), d(PSx_{n_i}, TTy_{n_i}), d(PSx_{n_i}, SSx_{n_i}), d(TTy_{n_i}, QTy_{n_i}))$$

for all $i \in \omega$. Hence the upper semicontinuity of φ yields

$$d(Sa, Tb) \le \varphi(d(Sa, Tb), d(Sa, Tb), d(Sa, Tb), 0, 0))$$

which implies $d(Sa, Tb) = 0$ or $Sa = Tb$. Similarly since

$$d(Pa, QTy_{n_i}) \to d(Pa, Tb), \quad d(Sa, TTy_{n_i}) \to d(Sa, Tb) = 0,$$
$$d(Sa, QTy_{n_i}) \to d(Sa, Tb) = 0, \quad d(Pa, TTy_{n_i}) \to d(Pa, Tb), \text{ and}$$
$$d(TTy_{n_i}, QTy_{n_i}) \to d(Tb, Tb) = 0$$

for some subsequence $\{n_i\}_{i\in\omega}$ of $\{n\}_{n\in\omega}$, it follows from

$$d(Pa,\ QTy_{n_i}) \le \varphi(d(Sa,\ TTy_{n_i}),\ d(Sa,\ QTy_{n_i}),\ d(Pa,\ TTy_{n_i}),$$
$$d(Pa,\ Sa),\ d(TTy_{n_i},\ QTy_{n_i}))$$

that

$$d(Pa,\ Tb) \le \varphi(0,\ 0,\ d(Pa,\ Tb),\ d(Pa,\ Tb),\ 0)$$

which shows $Pa = Tb$. Hence we have

$$d(Pa,\ Qb) \le \varphi(d(Sa,\ Tb),\ d(Sa,\ Qb),\ d(Pa,\ Tb),\ d(Pa,\ Sa),\ d(Tb,\ Qb))$$
$$= \varphi(0,\ d(Pa,\ Qb),\ 0,\ 0,\ d(Pa,\ Qb)),$$

and so $Pa = Qb$. Consequently $SQb = SPa = PSa = PQb$, and hence we have

$$d(PQb,\ Qb) \le \varphi(d(SQb,\ Tb),\ d(SQb,\ Qb),\ d(PQb,\ Tb),\ d(PQb,\ SQb),$$
$$d(Tb,\ Qb))$$
$$= \varphi(d(PQb,\ Qb),\ d(PQb,\ Qb),\ d(PQb,\ Qb),\ 0,\ 0)$$

proving $PQb = Qb$. Since $TPa = TQb = QTb = QPa$, we obtain

$$d(Pa,\ QPa) \le \varphi(d(Sa,\ TPa),\ d(Sa,\ QPa),\ d(Pa,\ TPa),\ d(Pa,\ Sa),$$
$$d(TPa,\ QPa))$$
$$= \varphi(d(Pa,\ QPa),\ d(Pa,\ QPa),\ d(Pa,\ QPa),\ 0,\ 0)$$

which yields $QPa = Pa$. Therefore we have

$$SQb = SPa = PSa = PQb = Qb \quad \text{and} \quad TQb = QTb = QPa = Pa = Qb,$$

and thus $Qb(= Pa = Tb = Sa)$ is a common fixed point of P, Q, S, and T. It will suffice only to prove that Qb is the *unique* common fixed point of P and S. To this end, let $x \in X$ be a common fixed point of P and S. Then

$$d(x,\ Qb) = d(Px,\ Qb) \le \varphi(d(x,\ Qb),\ d(x,\ Qb),\ d(x,\ Qb),\ 0,\ 0),$$

and therefore $x = Qb$.

Corollary 1. *Let P, Q and T be mappings of a nonempty* L-*space* $(X, \to)$ *into itself such that* $PT = TP$, $QT = TQ$, $P(X) \cup Q(X) \subset T(X)$ *and* T *is continuous. Suppose that* $(X, \to)$ *is d-complete for some continuous semimetric* *d on X. If there exists an* $\alpha \in [0, 1)$ *such that*

(*) $\qquad d(Px, Qy) \le \alpha \cdot \max\{d(Tx, Ty), d(Px, Tx), d(Ty, Qy)\}$

for all x, y *in* X, *then each of pairs* $\{P, T\}$ *and* $\{Q, T\}$ *has a unique common fixed point and these two points coincide.*

Proof. Let $x_0 \in X$. Then, since $P(X) \cup Q(X) \subset T(X)$, one can find a sequence $\{x_n\}_{n\in\omega}$ in X such that $Tx_{2n+1} = Px_{2n}$ and $Tx_{2n+2} = Qx_{2n+1}$ for all $n \in \omega$. By virtue of the inequality (*), it is not hard to show that

$$d(Tx_{n+1}, Tx_n) \le \alpha^n \cdot d(Tx_1, Tx_0)$$

for all $n \in \omega$ (see the proof of Theorem 1 of Singh [5]). Consequently we have $\sum_{i=0}^{\infty} d(Tx_{n+1}, Tx_n) < \infty$, and hence $\{Tx_n\}_{n\in\omega}$ converges to some a in X. Since $Px_{2n} \to a$, $Qx_{2n+1} \to a$, $Tx_{2n} \to a$, and $Tx_{2n+1} \to a$, putting

$$\varphi(t_1, t_2, t_3, t_4, t_5) = \alpha \cdot \max\{t_1, t_4, t_5\}$$

for each $(t_1, t_2, t_3, t_4, t_5) \in \mathbb{R}_+^5$, Theorem 1 can be applied to obtain the required conclusion.

We shall derive a strengthened version of Theorem 2 of Singh[5]:

Corollary 2. *Let* P, Q *and* T *be mappings of an* L-*space* $(X, \to)$ *into itself such that* $PT = TP$, $QT = TQ$, $P(X) \cup Q(X) \subset T^t(X)$, *and* T *is continuous, where* t *is a positive integer. Suppose that* $(X, \to)$ *is d-complete for some continuous semimetric* d *on* X. *If there exist positive integers* p, q *and an* $\alpha \in [0, 1)$ *such that*

$$d(Ux, Vy) \le \alpha \cdot \max\{d(Ux, Sx), d(Vy, Sy), d(Sx, Sy)\}$$

for all x, y *in* X, *where* $U = P^p$, $V = Q^q$ *and* $S = T^t$, *then each of pairs* $\{P, T\}$ *and* $\{Q, T\}$ *has a unique common fixed point and these two points coincide.*

Proof. Since $U(X) \cup V(X) \subset P(X) \cup Q(X) \subset T^t(X)$, Corollary 1 shows that U and S have a unique common fixed point $a \in X$ which is the unique common fixed point of V and S. Obviously Pa is a common fixed point of U and S. Hence we have $Pa = a$. Similarly $Qa = a$ and $Ta = a$. If $x \in X$ is a common fixed point of P and T, then x is also a common fixed point of U and S, and so $x = a$. Thus a is the unique common fixed point of P and T. A similar argument shows that a is the unique common fixed point of Q and T.

The following is the main result of Park [4].

Corollary 3. *Let f, g be commuting mappings of a metric space (X, d) into itself, and φ an upper semicontinuous mapping of $\mathbb{R}_+^5$ into $\mathbb{R}_+$ which is nondecreasing in each coordinate variable and satisfies the inequality $\varphi(t, t, t, t, t) < t$ for every $t > 0$. Suppose that the following conditions are satisfied*:

(1) $$d(f(x), f(y)) \le \varphi(d(g(x), g(y)), d(f(x), g(x)), d(f(y), g(y)), d(f(y), g(x)), d(f(x), g(y)))$$
for all x, y in X.

(2) *There exists a sequence $\{x_n\}_{n\in\omega}$ in X such that $g(x_{n+1}) = f(x_n)$ for all $n \in \omega$, $\sup\{d(g(x_i), g(x_j)) \mid i, j \in \omega\} < \infty$, and such that $\{x_n\}_{n\in\omega}$ has a subsequence converging to an $a \in X$ at which g is continuous.*

Then $g(a)$ is a unique common fixed point of f and g.

Proof. The same argument used in the proof of Theorem 1 of [3] establishes that $\{g(x_n)\}_{n\in\omega}$ is a Cauchy sequence in X. Hence $g(x_n) \to a$ and $f(x_n) \to a$, where $\to$ denotes the convergence with respect to the metric d. It is clear that d is a continuous o-metric on the L-space $(X, \to)$. Thus applying Theorem 1 to $P = Q = f$ and $S = T = g$, we have the desired conclusion.

We shall now derive a strengthened version of the essential part of the main theorem of Fisher [1].

Corollary 4. *Let* S *and* T *be continuous mappings of a nonempty complete metric space* (X, d) *into itself, and* A *a mapping of* X *into* $S(X) \cup T(X)$ *which commutes with* S *and* T. *If there exists an* $\alpha \in [0, 1)$ *such that*

$$(**) \qquad d(Ax, Ay) \le \alpha d(Sx, Ty)$$

for all x, y *in* X, *then each of pairs* $\{A, S\}$ *and* $\{A, T\}$ *has a unique common fixed point and these two points coincides.*

Proof. Let $x_0 \in X$. Then we can find a sequence $\{x_n\}_{n \in \omega}$ in X such that $Sx_{2n+1} = Ax_{2n}$ and $Tx_{2n+2} = Ax_{2n+1}$ for all $n \in \omega$. Using the inequality (**), it is easy to see that

$$d(Ax_{n+1}, Ax_n) \le \alpha^n d(Ax_1, Ax_0)$$

for all $n \in \omega$ (see the proof of Theorem 2 of Fisher [1]). It follows that $\{Ax_n\}_{n \in \omega}$ is a Cauchy sequence in X, and hence it converges to some $a \in X$. Then $Ax_{2n} \to a$, $Tx_{2n} \to a$, $Ax_{2n+1} \to a$, $Sx_{2n+1} \to a$, where $\to$ denotes the convergence with respect to the metric d. Since d is a continuous o-metric on the L-space $(X, \to)$, Theorem 1 yields the conclusion of the corollary.

The following result extends a theorem of Singh [5]:

Theorem 2. *Let* P, Q, S, *and* T *be mappings of a separated* L-*space* $(X, \to)$ *into itself such that* $PS = SP$ *and* $QT = TQ$, *and* d *a nonnegative real valued function on* $X \times X$ *such that* $d(x, x) = 0$ *for all* x *in* X.

If

$$(\dagger)\quad d(Px, Qy) < \max\{d(Sx, Ty), d(Sx, Qy), d(Px, Ty), d(Px, Sx), d(Ty, Qy)\}$$

whenever $Px \neq Qy$, *and if there exist sequences* $\{x_n\}_{n\in\omega}$ *and* $\{y_n\}_{n\in\omega}$ *in* X *and points* a, b *in* X *such that* P *and* S *are continuous at* a, Q *and* T *are continuous at* b, *and such that* $Px_n \to a$, $Sx_n \to a$, $Qy_n \to b$, *and* $Ty_n \to b$, *then* Pa $(= Qb = Sa = Tb)$ *is a unique common fixed point of* P *and* S, *and at the same time, it is a unique common fixed point of* Q *and* T (*hence* Pa *is a unique common fixed point of* P, Q, S, *and* T).

Proof. Since S is continuous at a, we can find a subsequence s of $\{x_n\}_{n\in\omega}$ for which $SP\mathrm{s} \to Sa$. Then since $S\mathrm{s} \to a$, the continuity of P at a implies the existence of a subsequence t of s such that $PS\mathrm{t} \to Pa$. But then $PS\mathrm{t} = SP\mathrm{t}$ is a subsequence of $SP\mathrm{s}$, and so $PS\mathrm{t} \to Sa$. Since the space is separated, it follows therefore that $Pa = Sa$. Similarly we have $Qb = Tb$. If $Pa \neq Qb$ then we have

$$\begin{aligned} d(Pa, Qb) &< \max\{d(Sa, Tb), d(Sa, Qb), d(Pa, Tb), d(Pa, Sa), d(Tb, Qb)\} \\ &= d(Pa, Qb), \end{aligned}$$

a contradiction. Hence $Pa = Qb$. Consequently $TPa = TQb = QTb = QPa$, and so $Pa \neq QPa$ implies

$$\begin{aligned} d(Pa, QPa) &< \max\{d(Sa, TPa), d(Sa, QPa), d(Pa, TPa), d(Pa, Sa), d(TPa, QPa)\} \\ &= d(Pa, QPa), \end{aligned}$$

a contradiction. Therefore $QPa = Pa$. Similarly, since $SQb = SPa = PSa = PQb$, if $PQb \neq Qb$ then we have also a contradiction

$$d(PQb, Qb) < \max\{d(SQb, Tb), d(SQb, Qb), d(PQb, Tb), d(PQb, SQb), d(Tb, Qb)\}$$

$$= d(PQb,\ Qb).$$

It follows that $PQb = Qb$. Hence $SPa = PSa = PQb = Qb = Pa$ and $TQb = QTb = QPa = Pa = Qb$. Thus Pa $(= Qb = Sa = Tb)$ is a common fixed point of P, Q, S, and T. If $x \in X \setminus \{Qb\}$ is a common fixed point of P and S, then we have

$$\begin{aligned} d(x,\ Qb) &= d(Px,\ Qb) \\ &< \max\{d(Sx,\ Tb),\ d(Sx,\ Qb),\ d(Px,\ Tb),\ d(Px,\ Sx),\ d(Tb,\ Qb)\} \\ &= d(x,\ Qb). \end{aligned}$$

This shows that Qb is the unique common fixed point of P and S. Similarly we can show that Pa is the unique common fixed point of Q and T.

As an immediate consequence of this theorem, we have the following result which slightly improves Theorem 1 of Singh [5].

Corollary 5. *Let P, Q, and T be continuous mappings of a nonempty separated* L*-space $(X, \to)$ into itself such that $PT = TP$, $QT = TQ$, $P(X) \cup Q(X) \subset T(X)$. Suppose that $(X, \to)$ is d-complete for some semimetric d on X. If there exists an $\alpha \in [0, 1)$ such that*

$$d(Px,\ Qy) < \alpha \cdot \max\{d(Tx,\ Ty),\ d(Px,\ Tx),\ d(Ty,\ Qy)\}$$

for all x, y in X, then each of pairs $\{P, T\}$ and $\{Q, T\}$ have a unique common fixed point and these two points coincide.

Proof. The same argument used in the proof of Corollary 1 ensures the existence of a sequence $\{x_n\}_{n\in\omega}$ in X and an $a \in X$ such that $Px_{2n} \to a$, $Qx_{2n+1} \to a$, $Tx_{2n} \to a$, and $Tx_{2n+1} \to a$. Thus Theorem 2 can be applied.

Remark. If $a = b$ in Theorem 2, then (†) can be replaced with the following inequality

$$
\begin{aligned}
d(Px, Qy) < \max\{ & d(Px, Qx), d(Px, Sx), d(Px, Tx), d(Px, Ty), \\
& d(Py, Sy), d(Qx, Qy), d(Qx, Tx), d(Qx, Ty), \\
& d(Qy, Ty), d(Sx, Px), d(Sx, Qx), d(Sx, Qy), \\
& d(Sy, Py), d(Sx, Tx), d(Sx, Ty), d(Tx, Qx), \\
& \qquad\qquad d(Tx, Qy), d(Tx, Ty) \}.
\end{aligned}
$$

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