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(1, 2)-Ideal Characterizations of Certain Classes of Semigroups

Dedicated to Professor Kiyoshi Iséki on his sixtieth birthday

By Sándor LAJOS*

Let S be a semigroup. A subsemigroup A of S is called a (1, 2)-*ideal* of S if $ASA^2 \subset A$ (cf. [3]). In a recent paper the author [6] obtained several characterizations of semigroups which are set-theoretical unions of groups in terms of their (1, 2)-ideals. For example, S is completely regular if and only if the set $12(S)$ of all (1, 2)-ideals of is a multiplicative band (cf. [8], Theorem 1). In this paper some further characterizations will be given by (1, 2)-ideals. For the terminology not defined here, see A. H. Clifford and G. B. Preston [1], or J. M. Howie [2].

Theorem 1. *For a semigroup S the following conditions are equivalent:*

(A) *S is a semilattice of groups.*

(B) *The set $12(S)$ of all (1, 2)-ideals of S is a distributive lattice with respect to set product and union.*

(C) *The set $12(S)$ is a lattice with respect to set product and union.*

(D) *The set $12(S)$ is a multiplicative semilattice.*

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Proof. (A) implies (B): Let S be a semigroup which is a semilattice of groups. Then every (1, 2)-ideal of S is a two-sided ideal (cf. [5]). Thus the set $12(S)$ coincides with the set of all two-sided ideals of S, which evidently is a distributive lattice with respect to set product and set union because of S is a regular semigroup.

Evidently (B) implies (C) implies (D).

(D) implies (A): If S is a semigroup with property that $12(S)$ is a multiplicative semilattice, then it follows that

$$L \cap R = (L \cap R)^2 \subset LR = RL \subset L \cap R \tag{1}$$

holds for every left ideal L and every right R of S. Hence we get $L \cap R = LR$ for any left ideal L and right ideal R of S, whence S is a semilattice of groups (cf. [4]).

Remark. Let m and n be positive integers. Then our Theorem 1 remains true with m, n instead of 1, 2.

Theorem 2. *A semigroup S is a semilattice of left groups if and omly if the set $12(S)$ of all (1, 2)-ideals of S is a multiplicative band and S is a right identity of it.*

Proof. If a semigroup S is a semilattice of left groups, then any (1, 2)-ideal of S is a right ideal of S. Thus $12(S)$ coincides with the set of all right ideals of S. But this is a multiplicative band and S is a right identity of it (cf. [9], Theorem 3).

Conversely, suppose that a semigroup S has the property that $12(S)$ is a band and S is a right identity of it. Then it follows that S is completely regular and $AS = A$ holds for every (1, 2)-ideal A of S. Hence S is a regular left duo semigroup, which is a semilattice of left

groups (cf. [10]).

Theorem 3. *A semigroup S is a semilattice of groups if and only if to every element a of S there exists an idempotent element $e = e(a)$ such that the principal (1, 2)-ideal generated by a,*

$$\langle a \rangle_{(1,\,2)} = eS = Se. \tag{2}$$

Proof. Let S be a semilattice of groups. Then $aS = Sa$ holds for every element a of S, i.e. the Green's relations $\mathcal{L}$ and $\mathcal{R}$ coincide on S. Hence we need to show that $\langle a \rangle_{(1,\,2)} = eSe$ holds for an idempotent element e of S. This follows at once from Theorem 2 in [6].

Conversely, assume that for every element a of a semigroup S there is an idempotent element $e = e(a)$ with property (2). Then it follows that $a = es = te$ with $s, t \in S$. Hence $ae = ea = a$. On the other hand, (2) implies $e = a$, $e = a^2$, or $e = aua^2$ where $u \in S^1$. Hence it follows that $a = a^2$, $a = a^3$, or $a = a^2(uau)a^2$ holds for any element a of S. Therefore S is completely regular.

Finally, (2) implies that every (1, 2)-ideal of S is a two-sided ideal of S because of every (1, 2)-ideal is the set-theoretical union of principal ones. Thus S is a completely regular duo semigroup which is a semilattice of groups.

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