On Some Problems on BCK-Algebras

Iséki, Kiyoshi

(Citation)
Mathematics Seminar Notes,7(1):173-178

(Issue Date)
1979

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/E0001468

(URL)
https://hdl.handle.net/20.500.14094/E0001468

MATHEMATICS SEMINAR NOTES
Vol. 7 (1979)

On Some Problems on BCK-Algebras

By Kiyoshi ISÉKI*

About ten years ago, we introduced a new notion which is called a BCK-algebra (for detail, see [2]).

Various authors have obtained interesting and important results about BCK-algebras, which have been published in these Notes and Mathematica Japonica. A list of publications on BCK-algebras and its related topics may be found in [6]; it includes the publications until 1977.

In this Note, we shall state some unsolved problems on BCK-algebras.

§ 1. On the definition of BCK-algebras.

H. Yutani [11] showed that the class of commutative BCK-algebras is an equational class (a variety). Other important classes, for instance, the classes of positive implicative BCK-algebras are also equational classes. A bounded BCK-algebra X is a BCK-algebra with largest element 1, i.e., $x \leq 1$ for every $x \in X$; then $1 * x$ will be denoted by Nx.

Let X be a bounded BCK-algebra with condition (S). As is well known,

$$(a * b) * c = a * (b \circ c)$$

holds for any BCK-algebra with condition (S), so we have

Received April 2, 1979.

* Department of Mathematics, Kobe University, Nada, Kobe, 657, Japan.

$$Nb * c = N(b \circ c).$$

If X satisfies $NNx = x$ for every $x \in X$, then

$$b \circ c = N(Nb * c).$$

Hence *the* $\circ$ *operation is definable from the operations* N *and* $*$. Therefore *the class of bounded, commutative BCK-algebra with condition* (S) *is an equational class*, as a commutative BCK-algebra with 1 satisfies $NNx = x$.

On classes of BCK-algebras, there are the following unsolved problems which are very important to develop the theory of BCK-algebras.

Problem 1. Is the class of BCK-algebra an equational class ?

Problem 2. Is the class of BCK-algebra with condition (S) an equational class ?

On the condition $NNx = x$, we have the following

Problem 3. Is there a non-commutative BCK-algebra satisfying $NNx = x$?

§ 2. On BCK-algebras satisfying condition (*): $a \le b$ implies $a \circ (b * a) = b$.

There is a commutative BCK-algebra with condition (S) not satisfying (*).

A BCK-algebra X is *positive implicative*, if X satisfies $(x * y) * = x * y$ for every $x, y \in X$. If a positive implicative BCK-algebra X is commutative, then it is called to be *implicative*.

If X is positive implicative, then X satisfies condition (*). But the converse is not always true.

Let X be a positive implicative BCK-algebra with condition (S). If $a \le b$, then, by Theorem 1 in [3],

$$a \circ (b * a) \le a \circ b \le b \circ b = b,$$

and

$$b * (a \circ (b * a)) = (b * a) * (b * a) = 0.$$

Hence X satisfies the condition (*).

On the other hand, $a \leq b$ implies

$$b \leq a \circ b \leq b \circ b = b,$$

which shows $a \circ b = b$.

Let x be an element with $a \circ x = b$. Then we have

$$(b * a) * x = b * (a \circ x) = b * b = 0.$$

Hence $b * a \leq x$. Moreover $x \leq b$. Therefore, $b * a$ is the smallest element satisfying $a \circ x = b$, and b is the largest element satisfying $a \circ x = b$ in a positive implicative BCK-algebra with condition (S).

From the above proof, we know that $b * a$ *is the smallest element satisfying* $a \circ x = b$ *under condition* (*).

If X is bounded, then for any element a, there is the smallest element x satisfying $a \circ x = 1$. The smallest element x is given by Nx.

The present author proved that any bounded, commutative BCK-algebra is a lattice with respect to $\wedge$, and $\vee$, where $a \vee b$ is defined by $N(Na \wedge Nb)$. Quite recently, T. Tracyzk [8] showed that any commutative BCK-algebra that any two elements have an upper bound is a distributive lattice. This result is a fundamental one on a commutative BCK-algebra. From this basic result, we have many important theorem.

As an easy result, we know that *any bounded, commutative BCK-algebra is a de Morgan algebra*. Further, *any commutative BCK-algebra with condition* (S) *is a distributive lattice with respect to the original order* $\leq$.

§ 3. 0-distributivity of a BCK-algebra.

It is easily seen that *any commutative BCK-algebra* X *with condition* (S) *is* 0-*distributive with respect to* $\wedge$, $\circ$, i.e. $a \wedge b = c$ and $a \wedge c = 0$ imply $a \wedge (b \circ c) = 0$.

To prove this, we first remark that $a \wedge b$ is given by $a * (a * b)$. Then we have

$$(a * ((a * b) * c)) * (a * (a * b)) \leq (a * b) * ((a * b) * c)$$
$$= (a * b) \wedge c \leq a \wedge c = 0.$$

Since $a \wedge b = 0$, we obtain $a * ((a * b) * c) = 0$. This implies $0 = a * (a * (b \circ c)) = a \wedge (b \circ c)$. This means that X is 0-distributive.

If a commutative BCK-algebra with condition (S) satisfies

$$a \wedge (b \circ c) = (a \wedge b) \circ (a \wedge c),$$

then it is positive implicative.

Let $a = b = c$ in the above relation. Then we have $a \wedge (a \circ a) = (a \wedge a) \circ (a \wedge a)$. Hence $a \circ a = a$, which means that the algebra is positive implicative (for more detail, see [7]).

The notion of 0-distributivity in a lattice was introduced by J. C. Varlet [9] as a generalization of the notion of pseudo-complementedness. It has been studied by J. C. Varlet [10] and W. H. Cornich [1].

We shall define a relation $\sim$ like in a lattice as follows. $a \sim b$ if and only if $a \wedge x = 0$ is equivalent to $b \wedge x = 0$ (see [9]).

Then $\sim$ is an equivalence relation on a commutative BCK-algebra with condition (S). Let $a \sim b$. Then $(a \wedge c) \wedge x = 0$ implies $a \wedge (c \wedge x) = 0$. Hence $b \wedge (c \wedge x) = 0$, which means $(b \wedge c) \wedge x = 0$. Therefore $a \sim b$ and $c \sim d$ imply $a \wedge c \sim b \wedge d$. Moreover, if $a \sim b$, and $(a \circ c) \wedge x = 0$, then, we have $a \wedge x = 0$, $c \wedge x = 0$. Hence $b \wedge x = 0$ and $c \wedge x = 0$, which implies $(b \circ c) \wedge x = 0$ by 0-distributivity. This shows that $a \sim b$ and $c \sim d$ imply $a \circ c \sim b \circ d$. Therefore $\sim$ is a congruence relation with respect to $\wedge$ and $\circ$.

If $a \wedge x = 0$, then by 0-distributivity, $(a \circ a) \wedge x = 0$. Hence $a \sim a \circ a$. In general we have $a \sim na\ (n = 2, 3, \cdots)$.

Next we shall consider the quotient algebra $X/\sim$. Then the class contatining 0 consists of the only element 0. We denote the class containing $a \in X$ by A. We define $A \wedge B$, $A \circ B$ by the classes containing $a \wedge b$, $a \circ b\ (a \in A,\ b \in B)$ respectively.

$X/\sim$ is a semilattice with respect to $\wedge$. Moreover it is semilattice with respect to $\circ$, as by $a \sim a \circ a\ (a \in A)$ we have $A \circ A = A$. We shall show $(a \wedge b) \circ a \sim a$, which means $(A \wedge B) \circ A = A$. Let $((a \wedge b) \circ a) \wedge x = 0$. Then $a \wedge x = 0$. Conversely, let $a \wedge x = 0$. Then $(a \wedge b) \wedge x = 0$. Hence, by 0-distributivity, we obtain $((a \wedge b) \circ a) \wedge x = 0$. Therefore $(a \wedge b) \circ a \sim a$.

Next we shall show $(a \circ b) \wedge a \sim a$, which means $(A \circ B) \wedge A = A$. Let $a \wedge x = 0$. Then $((a \circ b) \wedge a) \wedge x \leq a \wedge x = 0$. Let $((a \circ b) \wedge a) \wedge x = 0$. Since $a \leq a \circ b$, we have $a \wedge x = (a \wedge a) \wedge x \leq ((a \circ b) \wedge a) \wedge x = 0$. Hence $(a \circ b) \wedge a \sim a$.

From the above result, we know that $X/\sim$ *is a lattice.*

Moreover, we shall show $(a \circ b) \wedge c \sim (a \wedge c) \circ (b \wedge c)$, which means $(A \circ B) \wedge C = (A \wedge C) \circ (B \wedge C)$. Let $((a \circ b) \wedge c) \wedge x = 0$. Then $(a \wedge c) \wedge x = (b \wedge c) \wedge x = 0$. By 0-distributivity, we have $((a \wedge c) \circ (b \wedge c))\ \ x = 0$. Let $((a \wedge c) \circ (b \wedge c)) \wedge x = 0$. Then $(a \wedge c) \wedge x = (b \wedge c) \wedge x = 0$. Hence, by 0-distributivity, $a \wedge (c \wedge x) = b \wedge (c \wedge x) = 0$ implies $(a \circ b) \wedge (c \wedge x) = 0$, which is $((a \circ b) \wedge c) \wedge x = 0$. This shows that $X/\sim$ is a distributive lattice with respect to $\wedge$, $\circ$.

References

[1] W. H. Cornish: Quasicomplemented lattice, Comm. Math. Univ. Carolinae, 15 (1974), 501-511.

[2] K. Isêki: BCK-algebras, these NOTES, 4 (1976), 77-85.

[3] K. Isêki: On a positive implicative algebra with condition (S), these NOTES, 5 (1977), 227-232.

[4] K. Isêki: On bounded positive implicative algebras with condition (S), these NOTES, 5 (1977), 239-244.

[5] K. Isêki: On implicative BCK-algebra with condition (S), these NOTES, 5 (1977), 249-253.

[6] K. Isêki: Topics of BCK-algebras, Proc. 1st. Symposium on semigroups, Shimane Univ., 1977.

[7] K. Isêki: On commutative BCK-algebras with condition (S), these NOTES, 6 (1978), 137-138.

[8] T. Tracyzk: On the variety of bounded commutative BCK-algebras, to appear Math. Japonica.

[9] J. C. Varlet: A generalization of the notion of pseudo-complementedness, Bull. Soc. Roy. Sc. Liege, 3-4 (1960), 149-158.

[10] J. C. Varlet: Distributive semilattices and Boolean lattices, Bull. Soc. Roy. Sc. Liege, 41 (1972), 5-10.

[11] H. Yutani: The class of commutative BCK-algebras is equationally definable, these NOTES, 5 (1977), 207-210.