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Subba Rao, B. V.

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Lattice Ordered Autometrized Algebras II

By B. V. SUBBA RAO^{*}

Introduction. This paper is a continuation of [9], in which we introduced the notion of a representable autometrized algebra and proved the theorem: A lattice ordered autometrized algebra A is a subdirect product of totally ordered representable autometrized algebra if and only if A is representable and $(a * (a \cup b)) \cap (b * (a \cup b)) = 0$ for all a, b in A . In this paper, we study the geometry of a representable autometrized algebra, by imposing two more natural conditions (1) $(a * (a \cup b)) \cap (b * (a \cup b)) = 0$ and (2) $a * (a \cap b) + a \cap b = a$, on any such algebra. The class of representable autometrized algebras satisfying (1) and (2) is, still, a wider class than that of DR ℓ -semigroups in which $(a - b) \cap (b - a) \leq 0$ (see Example 1 below). We obtain almost all geometric results that were obtained by Rama Rao [8] for commutative Boolean- ℓ -algebras (which are commutative DR ℓ -semigroups in which $(a - b) \cap (b - a) \leq 0$). The geometric results obtained in this paper generalize the geometric results of Swamy [10], [11] for commutative ℓ -groups and those of Blumenthal [2], [3] and Ellis [4], [5] for Boolean algebras.

In this paper, we freely use the definition, results and notation of [9].

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^{*} Department of Mathematics, Andhra University, Waltair, India.

§ 1. We, first, recall the definition of representable autometrized algebra.

Definition 1. A lattice ordered autometrized algebra $A = (A, +, \leq, *)$ is called a *representable autometrized algebra* if and only if

(1.1) A is a semiregular autometrized algebra,

(1.2) For every a in A , all the mappings $x \mapsto a + x, x \mapsto a \cup x, x \mapsto a \cap x$ and $x \mapsto a * x$ are contractions. (a mapping $f: A \rightarrow A$ is called a *contraction mapping* with respect to a metric $*$ on A if and only if $f(x) * f(y) \leq x * y$ for all x, y in A)

Throughout this paper, $A = (A, +, \leq, *)$ stands for a representable autometrized algebra satisfying the conditions

(1.3) $(a * (a \cup b)) \cap (b * (a \cup b)) = 0$ for all a, b in A , and

(1.4) $(a * (a \cap b)) + (a \cap b) = a$ for all a, b in A .

Any commutative DR ℓ -semigroup $A = (A, +, \leq, -)$ which satisfies the condition $(a - b) \cap (b - a) \leq 0$ for all a, b in A , is a representable autometrized algebra satisfying (1.3) and (1.4). Consequently, Boolean algebras and commutative ℓ -groups are representable autometrized algebras satisfying (1.3) and (1.4). The following example shows that the class of representable autometrized algebras satisfying (1.3) and (1.4) is wider than the class of DR ℓ -semigroups, in which $(a - b) \cap (b - a) \leq 0$ for all a, b .

Example 1. Let $A = Z \cup \{u\}$ where Z is the set of all integers and u is an element which is not in Z . Define $+$ in A as follows:

$a + b =$ usual sum of a and b , for all a, b in Z ,

$a + u = u = u + a$ for all a in Z , and

$$u + u = u.$$

Define $\leq$ in A as follows: For the elements in Z , let $\leq$ be the usual ordering. Define $a < u$ for all a in Z . Define $*$ in A as follows: $a * b = |a - b|$ for all a, b in Z , $a * u = u = u * a$ for all a in Z , and $u * u = 0$. Then, it can be verified that $A = (A, +, \leq, *)$ is a representable autometrized algebra satisfying (1.3) and (1.4). But, A is not a DR ℓ -semigroup since there exists no least element x in A such that $x + u \geq a$ for any a in Z .

The following examples show that the conditions (1.3) and (1.4) are independent on a representable autometrized algebra.

Example 2. Let A be the lattice of all closed subsets of the space of real numbers with usual topology. Then A is a Brouwerian algebra and hence is a representable autometrized algebra. It can be verified that A satisfies (1.4). But A does not satisfy (1.3). For, let X and Y denote the closed intervals $[0, 2]$ and $[2, 3]$ respectively. Then $(X * (X \cup Y)) \cap (Y * (X \cup Y)) = \{2\} \neq \emptyset$ (where $\emptyset$ is the empty set).

Example 3. Let $A = Z \cup \{u, v\}$, where Z is the set of all integers and u and v are elements which are not in Z . Define $+$ in A as follows: $a + b =$ usual sum of a and b for all a, b in Z , $a + u = u = u + a$ for all a in Z , $a + v = v = v + a$ for all a in Z , $u + v = u = v + u$, $u + u = u$ and $v + v = v$. Define $\leq$ in A as follows: For the elements in Z , let $\leq$ be the usual ordering. Define $u < a < v$ for all a in Z . Define $*$ in A as follows: $a * b = |a - b|$ for all a, b in Z , $a * u = v = u * a$ for all a in Z , $a * v = v = v * a$ for all a in Z , $u * v = v = v * u$ and $u * u = 0 = v * v$. Then, it can be verified

that $A = (A, +, \leq, *)$ is a representable autometrized algebra satisfying (1.3). But A does not satisfy (1.4), because, $v * (v \cap u) + (v \cap u) = v * u + u = v + u + u \neq v$.

Now, we show that A turns out to be distributive lattice.

Theorem 1. $(A, \leq)$ is a distributive lattice.

We need the following

Lemma 1. $a * (a \cup b) = b * (a \cap b)$ for all a, b in A .

Proof. We have $a * (a \cup b) = (a \cup (a \cap b)) * (a \cup b) \leq (a \cap b) * b = (a \cap b) * ((a \cup b) \cap b) \leq a * (a \cup b)$. Hence $a * (a \cup b) = b * (a \cap b)$ for all a, b in A .

Proof of Theorem 1. Clearly $(A, \leq)$ is a lattice. In order to show that A is a distributive lattice, it is enough to prove that $a \cap x = a \cap y$ and $a \cup x = a \cup y$ implies $x = y$ for a, x, y in A (see p. 39 of [1]). So, assume that for a, x, y in A , $a \cap x = a \cap y$ and $a \cup x = a \cup y$. Then $x = x * (x \cap a) + x \cap a$ (by (1.4)) $= a * (a \cup x) + a \cap x = a * (a \cup y) + a \cap y = y * (y \cap a) + y \cap a = y$ (by (1.4)). This completes the proof.

Remark 1. One can observe that the above theorem is true even when $A = (A, +, \leq, *)$ is a lattice ordered autometrized algebra satisfying (1.4) and in which, for each a , the mappings $x \mapsto a \cup x$ and $x \mapsto a \cap x$ are contractions. Example 3 shows that there exists a representable autometrized algebra $A = (A, +, \leq, *)$ such that $(A, \leq)$ is a distributive lattice and A does not satisfy (1.4).

§ 2. In this article, we begin a study of the geometry of a representable autometrized algebra $A = (A, +, \leq, *)$ satisfying (1.3) and (1.4). Following the terminology introduced in [6], we introduce the following

Definition 2. A subset S of an autometrized algebra $A = (A, +, \leq, *)$ is said to be a *subgeometry* if and only if a and b belong to S implies $a * b$ belongs to S .

We shall treat any three distinct elements a, b, c of an autometrized algebra $A = (A, +, \leq, *)$ as the vertices of a triangle with sides $a * b$, $b * c$ and $c * a$ and we denote this triangle by $\Delta(a, b, c)$. A triangle $\Delta(a, b, c)$ is said to have *fixty* if and only if $a * b = c$, $b * c = a$ and $c * a = b$.

Swamy [10] proved that in a commutative ℓ -group, for a $\Delta(a, b, c)$, the following three conditions are equivalent.

- (i) $\Delta(a, b, c)$ has fixty.
- (ii) $a \cup b = b \cup c = c \cup a$ and $a \cap b \cap c = 0$.
- (iii) $\Delta(a * b, b * c, c * a)$ has fixty and the set $(0, a, b, c)$ is a subgeometry.

Rama Rao [8] proved that the conditions (i), (ii) and (iii) are equivalent for a $\Delta(a, b, c)$ in a commutative Boolean- ℓ -algebra $L = (L, \cup, \cap, -)$ (which is a DR ℓ -semigroup in which $(a - b) \cap (b - a) \leq 0$). We generalize this result, in the following theorem, to a representable autometrized algebra $A = (A, +, \leq, *)$ satisfying (1.3) and (1.4).

Theorem 2. *The following are equivalent in A .*

- (1) $\Delta(a, b, c)$ has fixty.
- (2) $a \cup b = b \cup c = c \cup a$ and $a \cap b \cap c = 0$.

(3) $\Delta(a * b, b * c, c * a)$ has fixity and the set $(0, a, b, c)$ is a subgeometry.

We need the following

Lemma 2. $a * b = (a \cup b) * (a \cap b)$ for all a, b in A .

Proof. We have $a * b \leq a * (a \cup b) + (a \cup b) * b = a * (a \cup b) + (a \cup b \cup b) * ((a \cap b) \cup b) \leq a * (a \cup b) + (a \cup b) * (a \cap b)$ (by (1.2)) and similarly $a * b = b * a \leq b * (a \cup b) + (a \cup b) * (a \cap b)$. Hence, $a * b \leq [a * (a \cup b) + (a \cup b) * (a \cap b)] \cap [b * (a \cup b) + (a \cup b) * (a \cap b)] = (a * (a \cup b)) \cap (b * (a \cup b)) + (a \cup b) * (a \cap b) = (a \cup b) * (a \cap b)$ (by (1.3)). Further, $(a \cup b) * (a \cap b) \leq (a \cap b) * a + a * (a \cap b) \leq (a \cup b) * a + a * b$ and similarly $(a \cup b) * (a \cap b) \leq (a \cup b) * b + a * b$. Hence $(a \cup b) * (a \cap b) \leq [(a \cup b) * a + a * b] \cap [(a \cup b) * b + a * b] = ((a \cup b) * a) \cap ((a \cup b) * b) + a * b = a * b$ (by (1.3)). Thus $(a \cup b) * (a \cap b) = a * b$ for all a, b in A .

Lemma 3. $a \cup b = a * b + a \cap b$ for all a, b in A .

Proof. We have $a \cup b = (a \cup b) * ((a \cup b) \cap (a \cap b)) + (a \cup b) \cap (a \cap b)$ (by (1.4)) $= (a \cup b) * (a \cap b) + a \cap b = a * b + a \cap b$ (by Lemma 2).

Proof of Theorem 2. We show that $(1) \Leftrightarrow (3)$ and $(1) \Leftrightarrow (2)$. The implication $(1) \Rightarrow (3)$ is obvious.

$(3) \Rightarrow (1)$: Assume that $\Delta(a * b, b * c, c * a)$ has fixity and $(0, a, b, c)$ is a subgeometry. Now, $a * b$ should be any of a, b, c . Let $a * b = a$. If $b * c = a$, then $a * c = (a * b) * (b * c) = a * a = 0$, so that $a = c$. So $b * c \neq a$. Hence $b * c = b$ or c . If $b * c = b$, then $a * c = (a * b) * (b * c) = a * b = a$, so that $b * c = (a * b) * (a * c) = a * a = 0$,

which implies $b = c$. Hence $b * c \neq b$. So, let $b * c = c$. Then $a * c$ should be any of a, b, c . If $a * c = a$, then $c = b * c = (a * b) * (a * c) = a * a = 0$, which implies $b = c = 0$. So, $a * c \neq a$. If $a * c = b$, then $a = a * b = (a * c) * (c * b) = b * c = c$. So $a * c \neq b$. If $a * c = c$, then $a = a * b = (a * c) * (c * b) = c * c = 0$, which implies $a = b = 0$. Thus $b * c \neq c$. Hence $a * b \neq a$. Similarly, we can show that $a * b$ different from b . Hence $a * b = c$. Similarly $b * c = a$ and $c * a = b$. Hence $\Delta(a, b, c)$ has fixty. Thus (3) implies (1). Hence (1) and (3) are equivalent.

(1) $\Rightarrow$ (2): Assume that $\Delta(a, b, c)$ has fixty. So, we have $a * b = c$, $b * c = a$ and $c * a = b$. Clearly $a > 0$, $b > 0$ and $c > 0$. Now, $a \cup b = a \cup b + a * b$ (by Lemma 3) $= c + a \cap b \geq c$ (since $a \cap b \geq 0$). Similarly $b \cup c \geq a$ and $c \cup a \geq b$. Hence $a \cup b = b \cup c = c \cup a$. Now, $a = b * c \leq b * (a \cup b) + (a \cup b) * c = b * (a \cup b) + (a \cup c) * c$ and $c = a * b \leq a * (a \cup c) + (a \cup c) * b = b * (a \cup b) + (a \cup c) * a$. Hence $a \cap c \leq [b * (a \cup b) + (a \cup c) * c] \cap [b * (a \cup b) + (a \cup c) * a] = b * (a \cup b) + ((a \cup c) * c) \cap ((a \cup c) * a) = b * (a \cup b)$ (by (1.3)). Further, $b = a * c \leq a * (a \cup b) + (a \cup b) * c = a * (a \cup b) + (b \cup c) * c$, and $c = a * b \leq a * (a \cup b) + (a \cup b) * b = a * (a \cup b) + (b \cup c) * b$. Hence $b \cap c \leq [a * (a \cup b) + (b \cup c) * c] \cap [a * (a \cup b) + (b \cup c) * b] = a * (a \cup b) + ((b \cup c) * c) \cap ((b \cup c) * b) = a * (a \cup b)$ (by (1.3)). Hence $a \cap b \cap c \leq (b * (a \cup b)) \cap (a * (a \cup b)) = 0$ (by (1.3)). Since $a \cap b \cap c \geq 0$, we have $a \cap b \cap c = 0$. Thus (1) implies (2).

(2) $\Rightarrow$ (1): Assume that $a \cup b = b \cup c = c \cup a$ and $a \cap b \cap c = 0$. Clearly $a \geq 0$, $b \geq 0$ and $c \geq 0$. So, we have, $c = c * 0$ (by (1.1)) $= c * (c \cap a \cap b) = (c \cap (a \cup c)) * (c \cap a \cap b) \leq (a \cup c) * (a \cap b)$ (by (1.2)) $= (a \cup b) * (a \cap b) = a * b$ (by Lemma 2). So, $c \leq a * b$. Now, $a * b \leq a * (a \cup c) + (a \cup c) * b =$

$(a \cup 0) * (a \cup c) + (a \cup b) * b \leq 0 * c + (a \cup b) * b$ (by (1.2)) $= c + (a \cup b) * b$
 (by (1.1)), and $a * b \leq a * (b \cup c) + (b \cup c) * b = a * (a \cup b) + (b \cup c) * (b \cup 0)$
 $\leq a * (a \cup b) + c * 0$ (by (1.2)) $= a * (a \cup b) + c$ (by (1.1)). Hence, $a * b \leq$
 $[c + (a \cup b) * b] \cap [a * (a \cup b) + c] = c + ((a \cup b) * b) \cap (a * (a \cup b)) = c$ (by
 (1.3)). Thus $a * b \leq c$. Hence $a * b = c$. Similarly, $b * c = a$ and $c * a$
 $= b$. Hence $\Delta(a, b, c)$ has fixty. Hence (2) implies (1). Thus (1) and
 (2) are also equivalent.

Rama Rao (See Corollary 10, p. 194 of [8]) has shown that if A is a
 commutative DR ℓ -semigroup in which $(a - b) \cap (b - a) \leq 0$, such that
 $a - a \cap b \cap c = a$ implies $a \cap b \cap c = 0$, then $\Delta(a, b, c)$ has fixty if and
 only if $a \cup b = b \cup c = c \cup a$ and $a \cap b \cap c = 0$. In fact, we now have the
 following

Corollary 1. *If A is a commutative DR ℓ -semigroup in which
 $(a - b) \cap (b - a) \leq 0$ for all a, b , then, $\Delta(a, b, c)$ has fixty if and only
 if $a \cup b = b \cup c = c \cup a$ and $a \cap b \cap c = 0$.*

Remark 2. The condition (1.3) is crucial in the hypothesis of Theorem
 2. For, in Example 2, A is a representable autometrized algebra which
 satisfies (1.4) but does not satisfy (1.3). Here, let $P = [0, 2]$, $Q =$
 $[1, 3]$ and $R = [0, 1] \cup [2, 3]$. Then, it is easy to observe that $\Delta(P, Q,$
 $R)$ has fixty while $P \cap Q \cap R \neq \emptyset$ (empty set).

In any Boolean geometry, since the metric operation is a group opera-
 tion, there do not exist any isosceles triangles. In commutative ℓ -groups
 also, Swamy [11] has shown that equilateral triangles do not exist.
 Rama Rao [8] has shown that in any commutative Boolean- ℓ -algebra, there do

not exist equilateral triangles. We now extend Rama Rao's result in the following way.

Theorem 3. *There exist no equilateral triangles in A .*

Proof. Assume that $a * b = b * c = c * a$. Now, $c = c * (c \cap a) + c \cap a$ (by (1.4)) $\leq c * a + a$ (by (1.2)) $= a * b + a$ and $c = c * (c \cap b) + c \cap b$ (by (1.4)) $\leq c * b + b$ (by (1.2)) $= a * b + b$. Hence $c \leq (a * b + a) \cap (a * b + b) = a * b + a \cap b = a \cup b$ (by Lemma 3). Thus $c \leq a \cup b$. Similarly, $b \leq a \cup c$ and $a \leq b \cup c$. Hence $a \cup b = b \cup c = c \cup a$. Now, $a * b \leq a * (a \cup c) + (a \cup c) * b = a * (a \cup c) + (a \cup b) * b$ and $a * b = b * c \leq b * (a \cup b) + (a \cup b) * c = b * (a \cup b) + (a \cup c) * c$. Hence $a * b \leq [a * (a \cup c) + (a \cup b) * b] \cap [b * (a \cup c) + (a \cup c) * c] = (a * (a \cup c)) \cap (c * (a \cup c)) + (a \cup b) * b = (a \cup b) * b$ (by (1.3)). Similarly, we can prove that $a * b \leq (a \cup b) * a$. Hence $a * b \leq ((a \cup b) * b) \cap ((a \cup b) * a) = 0$ (by (1.3)). So we have $a * b = b * c = c * a = 0$, which implies $a = b = c$. Hence the triangle, $\Delta(a, b, c)$ degenerates (i.e. a, b, c are not distinct).

Corollary 2. *There exist no equilateral triangles in a commutative DRl-semigroup which satisfies $(a - b) \cap (b - a) \leq 0$ for all a, b .*

Theorem 4. *A is a chain if and only if it is free of triangles with fixty.*

Proof. Assume that A is not a chain. So, there exist a, b in A such that $a \neq b$, $a \cup b \neq a$ and $a \cup b \neq b$. Then $(a \cup b) * a$, $(a \cup b) * b$ and $a * b$ are three distinct elements in A such that the triangle $\Delta((a \cup b) * a, (a \cup b) * b, a * b)$ has fixty (routine verification), contradicting the hypothesis. Hence if A is free of triangles with fixty,

then A is a chain. Conversely, if A is a chain, then by Theorem 2, no triangle can have fixty.

Corollary 3. *If A is a commutative $DR\ell$ -semigroup which satisfies $(a-b) \cap (b-a) \leq 0$ for all a, b in A , then, A is a chain if and only if it is free of triangles with fixty.*

Theorem 5. *If all triangles in A are isosceles, then A is a chain.*

Proof. For any a, b in A , consider the triangle $\Delta(a \cup b, a \cap b, a)$. By Lemmas 3 and 1 $(a \cup b) * (a \cap b) = (a \cup b) * a$ implies $a \cup b = a * b + a \cap b = (a \cup b) * a + a \cap b = b * (a \cap b) + a \cap b = b$ (by (1.4)). $(a \cup b) * (a \cap b) = (a \cap b) * a$ implies $a \cup b = a * b + a \cap b = (a \cup b) * (a \cap b) + (a \cap b) = a * (a \cap b) + a \cap b = a$. $(a \cup b) * a = (a \cap b) * a$ implies $b = b * (b \cap a) + b \cap a = a * (a \cup b) + a \cap b = a * (a \cap b) + a \cap b = a$. Hence the triangle $\Delta(a \cup b, a \cap b, a)$ degenerates (i.e., $a \cup b, a \cap b, a$ are not distinct) and hence A is a chain.

Remark 3. As any simply ordered commutative ℓ -group is a representable autometrized algebra satisfying (1.3) and (1.4), it follows that the converse of the above theorem is false.

Corollary 4. *If A is a commutative $DR\ell$ -semigroup such that $(a-b) \cap (b-a) \leq 0$ for all a, b in A and all triangles in it are isosceles, then A is a chain.*

§ 3. We shall begin with the following

Definition 3. An element x in A is said to *lie metrically between* a and b (in symbols $(a, x, b)M$) if and only if $a * x + x * b = a * b$.

This betweenness does not, in general, satisfy the special inner point property (β) : $(a, b, c)M$ and $(a, c, b)M$ if and only if $b = c$ (See [7]). But this being the metric betweenness, it has the other basic property symmetry in the outer points (α) : $(a, b, c)M$ implies $(c, b, a)M$.

Any betweenness relation R is said to have transitivity t_1 , if and only if $(a, b, c)R$ and $(a, d, b)R$ imply $(d, b, c)R$, and is said to have transitivity t_2 if and only if $(a, b, c)R$ and $(a, d, b)R$ imply $(a, d, c)R$. Now, we have the following

Lemma 4. In A , three points a, b, c fail to have the special inner point property, if a, b, c are the vertices of an isosceles triangle in which the sum of any two sides is equal to their union.

Proof. Assume that a, b, c are the vertices of an isosceles triangle, for which $(a * b) \cup (a * c) = a * b + a * c$, $(a * b) \cup (b * c) = a * b + b * c$, $(a * c) \cup (b * c) = a * c + b * c$, and $a * b = a * c$. Then, $a * b + b * c = (a * b) \cup (b * c) \leq (a * b) \cup (b * a + a * c) = (a * b) \cup (b * a) \cup (a * c) = a * c$. Hence $a * b + b * c = a * c$. So, $a * c + c * b = a * b + c * b = a * c = a * b$. Thus, we have $(a, b, c)M$ and $(a, c, b)M$. But $b \neq c$, since a, b, c are the vertices of a triangle.

Lemma 5. If the metric betweenness has transitivity t_1 , then, A is free of isosceles triangles for which the sum of any two sides is equal to their union.

Proof. If $\Delta(a, b, c)$ is such an isosceles triangle, where $a * b = a * c$, then $(a, b, c)M$ and $(a, c, b)M$ hold. Hence by the transitivity t_1 follows that $(c, b, c)M$, i.e., $c = b$. Since a, b, c are the ver-

tices of a triangle, $c \neq b$.

Lemma 6. *In A , the metric betweenness has transitivity t_2 .*

Proof. $(a, b, c)M$ and $(a, d, b)M$ imply $a*b + b*c = a*c$ and $a*d + d*b = a*b$ which yield $a*d + d*b + b*c = a*b + b*c = a*c$, and hence $a*c \geq a*d + d*c$. But $a*c \leq a*d + d*c$. Thus $(a, d, c)M$.

Remark 4. Lemmas 4, 5 and 6 true even when A is simply a lattice ordered autometrized algebra.

Since A is a distributive lattice (by Theorem 1), there is already the notion of lattice betweenness [7], according to which $(a, b, c)L$ if and only if $a \wedge c \leq b \leq a \vee c$.

In Boolean algebras and commutative ℓ -groups, lattice betweenness and metric betweenness are equivalent. In general, it is not the case. But, we have the following

Theorem 6. *In A , lattice betweenness implies metric betweenness.*

We need the following Lemmas.

Lemma 7. $a*b + b*c = (a \wedge c)*b + b*(a \vee c)$ for all a, b, c in A .

Proof. Using Lemma 1 we have $a*b + b*c \leq a*(a \vee c) + (a \vee c)*b + b*(a \wedge c) + (a \wedge c)*c = [(a \vee c)*b + b*(a \wedge c)] + [a*(a \vee c) + a*(a \vee c)] = [(a \vee c)*b + b*(a \wedge c)] + 2(a*(a \vee c))$, and similarly, $a*b + b*c \leq [(a \vee c)*b + b*(a \wedge c)] + 2(c*(a \vee c))$. Hence $a*b + b*c \leq [((a \vee c)*b + b*(a \wedge c)) + 2(a*(a \vee c))] \wedge [((a \vee c)*b + b*(a \wedge c)) + 2(c*(a \vee c))] = [(a \vee c)*b + b*(a \wedge c)] + 2(a*(a \vee c)) \wedge 2(c*(a \vee c)) = (a \vee c)*b + b*(a \wedge c)$

(since in any lattice ordered semigroup $a \cap x = 0$ and $a \cap y = 0$ imply $a \cap (x + y) = 0$). Further, $(a \cup c) * b + b * (a \cap c) \leq (a \cup c) * a + a * b + b * c + c * (a \cap c) + (a * b + b * c) + 2(a * (a \cup c))$ (by Lemma 1), and similarly $(a \cup c) * b + b * (a \cap c) \leq (a * b + b * c) + 2(c * (a \cup c))$. Hence $(a \cup c) * b + b * (a \cap c) \leq [(a * b + b * c) + 2(a * (a \cup c))] \cap [(a * b + b * c) + 2(c * (a \cup c))]$
 $= (a * b + b * c) + 2(a * (a \cup c)) \cap 2(c * (a \cup c)) = a * b + b * c$ (by (1.3)).
Hence, we have $a * b + b * c = (a \cap c) * b + b * (a \cup c)$ for all a, b, c in A .

Corollary 5. In A , $(a, b, c)M$ if and only if $(a \cap c, b, a \cup c)M$.

Proof. This follows from Lemmas 2 and 7.

Lemma 8. In A , $a \leq b \leq c$ implies $(a, b, c)M$.

Proof. Assume that $a \leq b \leq c$. Then, $c * b = [c * (c \cap a) + c \cap a] * [b * (b \cap a) + b \cap a] = (c * a + a) * (b * a + a) \leq (c * a) * (b * a)$ (by (1.2)).
But $(c * a) * (b * a) \leq c * b$ (by (1.2)). Hence, $c * a = (c * a) * (b * a)$. And, $b * a \leq c * a$ since $b * a = (b \cap c) * (a \cap b) \leq c * a$ (by (1.2)). So, $c * b + b * a = (c * a) * (b * a) + (b * a) = (c * a) * ((c * a) \cap (b * a)) + (c * a) \cap (b * a) = c * a$ (by (1.4)).

Proof of Theorem 6. Assume that $(a, b, c)L$. So, $a \cap c \leq b \leq a \cup c$.
Hence $(a \cap c) * b + b * (a \cup c) = (a \cap c) * (a \cup c)$ (by Lemma 8). This implies $a * b + b * c = a * c$ (by Lemma 2 and 7). Hence $(a, b, c)M$.

Corollary 6. In a commutative DRL-semigroup satisfying $(a - b) \cap (b - a) \leq 0$, lattice betweenness implies metric betweenness.

Lattice betweenness and metric betweenness are not equivalent in A .

However, we have the following

Theorem 7. *In A , lattice betweenness and metric betweenness are equivalent if and only if metric betweenness has transitivity t_1 .*

The proof depends on the following theorem which we collect from [7].

Lemma 9 (Pitcher and Smiley). *If L is a lattice and R is a triadic relation defined for all triples of elements of L , then R is a lattice betweenness provided that the following conditions hold.*

- (i) R satisfies the postulates
 - (α): $(a, b, c)R$ implies $(c, b, a)R$
 - (β): $(a, b, c)R$ and $(a, c, b)R$ if and only if $b = c$;
- (ii) R has the transitivity t_1 ;
- (iii) $a \leq b \leq c$ implies $(a, b, c)R$;
- (iv) $(a, a \vee c, c)R$ and $(a, a \wedge c, c)R$ hold for every a, c ; and
- (v) If $(a, b, c)L$, then, in the sublattice generated by (a, b, c) the transitivity t_2 holds for R .

Proof of Theorem 7. Assume that metric betweenness has transitivity t_1 . It is enough if we verify (i) through (v) of Lemma 9 for the metric betweenness. Clearly metric betweenness satisfies (α) and (ii). Since the metric betweenness has transitivity t_1 , (β) holds, because, $(a, b, c)M$ and $(a, c, b)M$ implies $(c, b, c)M$, i.e., $c = b$. By Lemma 8, (iii) is also satisfied. Since we have $(a \wedge c, a \vee c, a \vee c)M$ and $(a \wedge c, a \wedge c, a \vee c)M$, by Corollary 4, (iv) is also satisfied. By Lemma 6, (v) is also satisfied. Hence metric betweenness and lattice betweenness are equivalent. Conversely, assume that lattice betweenness and metric

betweenness are equivalent. Since the lattice betweenness has transitivity t_1 in any lattice (See [7]), it follows that metric betweenness has the transitivity t_1 .

Corollary 7. *In a commutative DRL-semigroup satisfying $(a - b) \cap (b - a) \leq 0$ for all a, b , lattice betweenness and metric betweenness are equivalent if and only if metric betweenness has transitivity t_1 .*

Blumenthal [2] introduced the notion of B -linearity and D -linearity in a Boolean geometry: An n -tuple ($n \geq 3$) of distinct points of A is said to be B -linear if and only if there exists a labelling $(p_1, p_2, \dots, p_n)$ of its elements such that $B(p_i, p_j, p_k)$ holds whenever $1 \leq i < j < k \leq n$; and is said to be D -linear if and only if there exists a labelling $(p_1, p_2, \dots, p_n)$ of its elements such that $d(p_1, p_n) = \sum_{i=1}^{n-1} d(p_i, p_{i+1})$, where $d(p_i, p_j)$ is the metric operation and $B(p_i, p_j, p_k)$ means that p_j lies between p_i and p_k (lattice betweenness). There is no difficulty in extending this definition to a representable autometrized algebra and we have the following

Theorem 8. *B -linearity implies D -linearity, in A .*

Proof. Assume that an n -tuple P is B -linear. Then, there exists a labelling $(p_1, p_2, \dots, p_n)$ of elements of P such that $B(p_i, p_j, p_k)$ holds whenever $1 \leq i < j < k \leq n$, from which follows, in particular $B(p_i, p_{i+1}, p_n)$ ($i < n-1$) holds so that $(p_i, p_{i+1}, p_n)M$ (by Theorem 6), which implies $p_i * p_n = p_i * p_{i+1} + p_{i+1} * p_n$ for $i < n-1$. Hence $p_1 * p_n = p_1 * p_{2n-1} + p_2 * p_n = p_1 * p_2 + p_2 * p_3 + p_3 * p_n = \dots = \sum_{i=1}^{n-1} (p_i * p_{i+1})$, so that P is D -linear.

§ 4. Throughout this section, $A = (A, +, \leq, *)$ stands for a representable autometrized algebra satisfying (1.3), (1.4) and (1.5): $a \geq b \cup c$ and $a * b \geq a * c$ implies $b \leq c$. Now, we have the following

Theorem 9. *Lattice betweenness and metric betweenness are equivalent in A .*

Proof. Lattice betweenness implies metric betweenness in A (by Theorem 6). Now, assume $(a, b, c)M$. We may assume that $a \leq c$ (by virtue of Corollary 4). So, $b = b * (b \cap a) + b \cap a$ (by (1.4)) $\leq b * a + a$ (by (1.2)) $\leq c * a + a$ (since $(a, b, c)M$ implies $a * b + b * c = a * c$) $= c * (c \cap a) + c \cap a = c$ (by (1.4)). Thus $a \leq c$ and $b \leq c$ and hence $c \leq a \cup b$. But $c * a \geq c * b$. Hence by (1.5) it follows that $a \leq b$. Thus $a \leq b \leq c$. Hence $(a, b, c)L$.

Corollary 8. *Lattice betweenness and metric betweenness are equivalent in a commutative ℓ -group.*

Corollary 9. *Lattice betweenness and metric betweenness are equivalent in a Boolean geometry.*

Corollary 10. *Metric Betweenness has transitivity t_1 in A .*

Proof. This follows from Theorems 7 and 9.

Blumenthal [3] introduced the notion of a metrically convex Boolean algebra and following him, Swamy [10] introduced a metrically convex Boolean ℓ -group. Following them, we introduce the following

Definition 4. A representable autometrized algebra is said to be

metrically convex if and only if given $a \neq b$, there is an x such that $a \neq x \neq b$ and $a * x + x * b = a * b$. We say that an element c in A ($0 \neq c$) is an *atom* if and only if $0 < c$ and $0 < x < c$ is impossible for any x .

Now, we have the following algebraic characterization.

Theorem 10. *If A is totally ordered, then A is metrically convex if and only if A is atomfree.*

Proof. Let A be totally ordered. Assume that A is metrically convex and $0 < c$. Then, there is an x such that $0 \neq x \neq c$ and $(0, x, c)M$ which implies $(0, x, c)L$ (by Theorem 9). Hence $0 < x < c$. So, c is not an atom and hence A is atomfree. Conversely, assume that A is atomfree. Let $a \neq b$. We may assume that $a < b$. Then, there is an x such that $0 < x < a * b$ and so, $a \leq a + x \leq a + a * b = a \cap b + (a \cap b) * b = b$ (by (1.4)). Put $a + x = y$. If $y = a$, then $(a * b) * 0 = a * b = (a + x) * (b * a + a) \leq x * (b * a) = (a * b) * x$. But $a * b > x \cup 0$. So, by (1.5) it follows that $x \leq 0$, contradicting $x > 0$. Hence $a < a + x$. If $y = b$, then $a * b = a * (a + x) \leq 0 * x$ (by (1.2)) $= x$ (by (1.1)), which contradicts $a * b > x$. Hence $y < b$. Thus, $a < y < b$. Hence $(a, y, b)M$ (by Theorem 6) and $a \neq y \neq b$. Hence A is metrically convex.

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