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(Citation)

Mathematics Seminar Notes, 7(2):211-233

(Issue Date)

1979

(Resource Type)

journal article

(Version)

Version of Record

(JaLCD0I)

<https://doi.org/10.24546/E0001473>

(URL)

<https://hdl.handle.net/20.500.14094/E0001473>



Distributive Semilattices

By U. Maddana SWAMY^{*}

Distributive semilattices were first introduced by Grätzer and Schmidt (see [5], [6], [7] and [15]). In the present paper, we introduce the concept of an ideal in a (meet) semilattice as a directed (above) initial segment which is motivated towards the result that a subset P of a lattice L is a prime ideal if and only if its set theoretic complement is a prime filter. Previously Frink [4] and Venkatanarasimhan [19], [20] have introduced different concepts of an ideal in a semilattice (poset) which are easily seen to be different from the present one. Also, our definition coincides with the usual ideal concept in lattices and extends almost all known results on ideals of a distributive lattice to the case of a distributive semilattice. The proofs in most cases are not routine, though not difficult, since the ideals of a semilattice don't form a lattice in general under set inclusion; in fact, the ideals of a semilattice S form a lattice under set inclusion if and only if S is a lattice under the partial order induced by the semilattice operation on S .

A major part of this paper is a base for the investigations undertaken by the author [9]-[11] on the representation of distributive semilattices

Received November 17, 1978.

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by means of continuous sections of sheaves. In §1, we define ideals and filters in a semilattice and give a few necessary and sufficient conditions in terms of ideals and filters for a semilattice to be distributive. In §2, we study the distributive semilattices by means of prime ideals and prime filters and §3 is devoted to exhibit a distributive and permutable sublattice of the lattice of all congruences on a distributive semilattice which plays a vital role in proving a 'Grothendieck type' of representation theorem for distributive semilattices by sections of sheaves [10]. In §4, the space of prime filters of a distributive semilattice with the hull-kernel topology has been studied.

Throughout this paper, S denotes a meet semilattices $(S, \wedge)$ and $\leq$ denotes the partial order on S induced by the binary operation $\wedge$. For elementary results on partially ordered sets, semilattices and lattices we refer to [2].

§ 1. Semilattices. We begin with the following

Definition 1.1. A subset T of S is said to be *directed above* (*below*) if, for any a and $b \in T$, there exists $c \in T$ such that $a \leq c$ and $b \leq c$ ($c \leq a$ and $c \leq b$). A subset I of S is said to be an *initial* (*final*) *segment* if, for any $a \in I$ and $x \in S$, $x \leq a$ ($a \leq x$) implies $x \in I$. An initial (final) segment is said to be an *ideal* (*filter*) of S if it is directed above (below).

Note that S is an ideal of itself if and only if S is directed above and S is always a filter of itself. A final segment F of S is a filter of S if and only if F is a subsemilattice of S . Let $F(S)$ and $I(S)$ denote respectively the set of all filters of S and the set

of all ideals of S . For any $a \in S$, let $[a]$ denote the ideal $\{x \in S \mid x \leq a\}$ and $\langle a \rangle$ denote the filter $\{x \in S \mid a \leq x\}$. Clearly, $a \mapsto \langle a \rangle$ is an anti-monomorphism of S into $F(S)$ and hence we can consider S as a join subsemilattice of $F(S)$. The following propositions are direct verifications (also see [12]).

Proposition 1.2. $F(S)$ is closed under arbitrary intersections and has smallest and greatest elements and hence $F(S)$ is a complete lattice under set inclusion. Further, for any $T \subset S$, the smallest filter of S containing T is given by $\langle T \rangle = \{x \in S \mid \bigwedge_{i=1}^n a_i \leq x \text{ for some } a_1, \dots, a_n \in T\}$.

For any filters F and G , the join $[F \cup G]$ of F and G in the lattice $F(S)$ will be denoted by $F \vee G$.

Proposition 1.3. S has a largest (smallest) element if and only if the intersection of all non-empty filters (ideals) is again non-empty.

Proposition 1.4. $I(S)$ is closed under finite intersections.

We observe that $I(S)$ need not be closed under arbitrary intersection. In fact, we have following

Theorem 1.5. The following are equivalent in any semilattice S :

- (i) $(S, \leq)$ is a lattice.
- (ii) $I(S)$ is closed under arbitrary intersections.
- (iii) For any $F \in F(S)$, $\bigcap_{a \in F} [a]$ is an ideal of S .
- (iv) $I(S)$ is a lattice under set inclusion.

Proof. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are clear. If we assume (iii) and I and $J \in I(S)$, then $\bigcup_{\substack{a \in I \\ b \in J}} \{x \in S \mid x \leq y \text{ for all } a \in I, b \in J\}$

$y \in [a] \cap [b]$ is the smallest ideal of S containing I and J , and hence (iii) implies (iv) (observe that, since the empty set is a filter of S , (iii) implies that S is an ideal of itself and hence directed above). Now, assume (iv) and let a and $b \in S$. Let I be the least upper bound of $[a]$ and $[b]$ in $I(S)$. Then, there exists $c \in I$ such that $a \leq c$ and $b \leq c$ and hence $I = [c]$. It follows that c is the least upper bound of a and b in S . Hence (iv) implies (i).

Definition 1.6. An ideal (filter) P of S is said to *proper* if $P \neq S$ and $P \neq \emptyset$. A proper ideal (filter) P of S is said to be *prime* if $S - P$ is directed below (above). A proper ideal (filter) of S is said to be *maximal* if it is not contained in any proper ideal (filter) of S other than itself.

Lemma 1.7. Let $P \subset S$. Then P is a prime ideal (filter) of S if and only if $S - P$ is a prime filter (ideal) of S .

Lemma 1.8. An ideal (filter) P of S is prime if and only if $P \neq S$ and $P \neq \emptyset$ and, for any ideals (filters) I and J of S , $I \cap J \subset P$ implies $I \subset P$ or $J \subset P$.

Definition 1.9. (Grätzer and Schmidt). S is said to be *distributive* if, for any a, b and $c \in S$ such that $a \wedge b \leq c$, there exists x and $y \in S$ such that $a \leq x$, $b \leq y$ and $x \wedge y = c$.

If S is distributive, then it can be easily seen that S is directed above and hence a finite distributive meet semilattice is a lattice. For any ideal I of S and a and $b \in S$, let

$$\langle a, I \rangle^* = \{x \in S \mid x \wedge a \in I\}$$

$$\text{and } \langle a, b \rangle = \{x \in S \mid x \wedge a = x \wedge b\}.$$

If $I = [b]$, we simply write $\langle a, b \rangle^*$ for $\langle a, [b] \rangle^*$. We now prove the following

Theorem 1.10. *The following are equivalent in any semilattice S :*

- (i) S is distributive.
- (ii) $F(S)$ is a distributive lattice.
- (iii) S generates a distributive sublattice in $F(S)$.
- (iv) For any ideal I of S and $a \in S$, $\langle a, I \rangle^*$ is an ideal of S .
- (v) For any a and $b \in S$, $\langle a, b \rangle^*$ is an ideal of S .
- (vi) For any a and $b \in S$, $\langle a, b \rangle$ is an ideal of S .
- (vii) Every filter which is maximal w.r.t. not containing an element is prime.
- (viii) Every filter F of S is the intersection of all prime filters of S containing F .

Proof. The equivalence of (i) and (ii) is established by Schmidt and the implication (ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (iv): Let $a \in S$ and I an ideal of S . Clearly $\langle a, I \rangle^*$ is an initial segment of S . Let x and $y \in S$, such that $x \wedge a$ and $y \wedge a \in I$. There exists $b \in I$ such that $x \wedge a \leq b$ and $y \wedge a \leq b$, so that

$$b \in ([x] \vee [a]) \cap ([y] \vee [a]) = ([x] \cap [y]) \vee [a]$$

and therefore there exists $z \in S$ such that $x \leq z$, $y \leq z$, and $z \wedge a \leq b$ which implies that $z \in \langle a, I \rangle^*$. Hence $\langle a, I \rangle^*$ is an ideal of S .

The implication (iv) $\Rightarrow$ (v) is trivial and the equivalence of (v) and (vi) follows from Proposition 1.4 and from the facts that, for any a and $b \in S$, $\langle a, b \rangle = \langle a, b \rangle^* \cap \langle b, a \rangle^*$ and $\langle a, b \rangle^* = \langle a \wedge b, a \rangle$.

(vi) $\Rightarrow$ (vii): Let F be a filter of S which is maximal w.r.t. not containing an element a of S . Let x and $y \in S - F$. Then $a \in (F \vee [x]) \cap (F \vee [y])$ and therefore there exists $b \in F$ such that $b \wedge x \leq a$ and $b \wedge y \leq a$ and hence x and $y \in \langle b, a \rangle^*$ which is an ideal of S . It follows that there exists $z \in \langle b, a \rangle^*$ such that $x \leq z$ and $y \leq z$. Now, $z \in b \leq a$ and therefore $z \in S - F$. Hence $S - F$ is directed above and F is prime.

The implication (vii) $\Rightarrow$ (viii) is obvious.

(viii) $\Rightarrow$ (ii): Let I, J and K be filters of S . For any prime filter F of S , we have $(I \cap J) \vee (I \cap K) \subset F \Leftrightarrow I \cap J \subset F$ and $I \cap K \subset F \Leftrightarrow I \subset F$ or $J \vee K \subset F \Leftrightarrow I \cap (J \vee K) \subset F$, from which it follows that $(I \cap J) \vee (I \cap K) = I \cap (J \vee K)$.

Remark 1.11. A part of the above theorem has been generated recently to partially ordered sets by the author and Krishna Murty [12]. In [1], Balbes defined a prime semilattice and obtained a necessary and sufficient condition which is analogous to (vii) above, for a semilattice to be a prime semilattice. In the case of lattices, the equivalence of (i) and (v) has been proved by Mandelkar [13].

Corollary 1.12. *If S is a distributive semilattice with smallest element 0 , then, for any $a \in S$, $(a)^* = \langle a, 0 \rangle^* = \{x \in S \mid a \wedge x = 0\}$ is an ideal of S .*

Corollary 1.13 (E. T. Schmidt [15]). *If S is an implicative semilattice [14], then S is distributive.*

§ 2. Ideals and filters in distributive semilattices. In this section,

we prove some preliminary results on distributive semilattices, in particular, on prime ideals and prime filters. The interesting factor is that almost all results of prime ideals of a distributive lattice are valid in distributive semilattices also.

Throughout this section, let S denote a distributive meet semilattice, unless otherwise stated. If L is a subsemilattice of S , then L need not be distributive. For example, if S is the semilattice of all positive integers, where, for any a and $b \in S$, $a \wedge b$ is the greatest common divisor of a and b , and if L is the subsemilattice consisting of all prime powers, then L is not distributive, even though S is distributive. It follows that the class of all distributive semilattices is not equational. However, we have the following

Proposition 2.1. *Let I be an ideal of S and J a filter of S such that $I \cap J \neq 0$. Then, $I \cap J$ is a distributive subsemilattice of S .*

Proof. Clearly $I \cap J$ is a subsemilattices of S . Let a, b and $c \in I \cap J$ such that $a \wedge b \leq c$. There exist x and $y \in S$ such that $a \leq x$, $b \leq y$ and $x \wedge y = c$. Since I is an ideal of S , there exists $z \in I$ such that $a \leq z$, $b \leq z$ and $c \leq z$. Now, we have, $c = c \wedge z = (x \wedge y) \wedge (y \wedge z)$, $a \leq x \wedge z \in I \cap J$ and $b \leq y \wedge z \in I \cap J$. Hence $I \cap J$ is distributive.

Corollary 2.2. *Every non-empty ideal (filter) of S is a distributive subsemilattice of S .*

Corollary 2.3. *For any a and $b \in S$ such that $a \leq b$, $[a, b] = \{x \in S \mid a \leq x \leq b\}$ is a distributive subsemilattice of S .*

For any a and $b \in S$, if the least upper bound of a and b exists

in S , then we say that $a \vee b$ exists in S . The following proposition follows from (v) of Theorem 1.10.

Proposition 2.4 (Balbes [1]). *Let b and $a_1, a_2, \dots, a_n \in S$. Then,*

- (i) *if $a_1 \vee a_2 \vee \dots \vee a_n$ exists in S , then $(a_1 \wedge b) \vee \dots \vee (a_n \wedge b)$ exists in S and is equal to $(a_1 \wedge a_2 \vee \dots \vee a_n) \wedge b$.*
- (ii) *if $a_i \vee b$ exists in S for all i , then $(a_1 \wedge a_2 \wedge \dots \wedge a_n) \vee b$ exists in S and is equal to $(a_1 \vee b) \wedge (a_2 \vee b) \wedge \dots \wedge (a_n \vee b)$.*

Corollary 2.5. *Let $(L, \wedge, \vee)$ be a lattice. Then the following are equivalent:*

- (i) *$(L, \wedge, \vee)$ is a distributive lattice.*
- (ii) *$(L, \wedge)$ is a distributive meet semilattice.*
- (iii) *$(L, \vee)$ is a distributive join semilattice.*

The following theorem and its proof are analogous to that of a well known Stone's theorem. A similar result for semigroups was proved by Kist (Lemma 1.2 of [8]) in which the definition of an ideal is different from our definition.

Theorem 2.6. *Let I be an ideal of S and L a subsemilattice of S such that $I \cap L = \emptyset$. Then, there exists a prime ideal P of S such that $I \subset P$ and $P \cap L = \emptyset$.*

Proof. Let $T = \{x \in S \mid a \leq x \text{ for some } a \in L\}$. Then T is a filter of S and $T \cap I = \emptyset$ and hence, by Zorn's lemma, there exists a filter F of S which is maximal among all filters containing L , and disjoint from I . Now, $I \subset S - F$ and $(S - F) \cap L = \emptyset$. By Lemma 1.7, it is enough if we prove that F is a prime filter. Let x and $y \in S - F$. By maximality of

F , there exist $a \in I \cap (F \vee [x])$ and $b \in I \cap (F \vee [y])$ and therefore there exists $c \in I$ such that $a \leq c$ and $b \leq c$. Hence $c \in I \cap (F \vee ([x] \cap [y])) = I \cap (F \vee ([x] \cap [y]))$ which implies that $[x] \cap [y] \notin F$. Thus F is a prime filter of S .

Remark 2.7. The converse of the above theorem is also true: i.e., if S is any semilattice in which, for any ideal I and any subsemilattice L such that $I \cap L = \emptyset$, there exists a prime ideal P of S such that $I \subset P$ and $P \cap L = \emptyset$, then S is a distributive semilattice. This follows from Theorem 1.10 and Lemma 1.7.

Corollary 2.8. *Let L be a subsemilattice of S and P a prime ideal (filter) of L . Then, there exists a prime ideal (filter) Q of S such that $Q \cap L = P$.*

Proof. Let P be a prime ideal of L and $P_1 = \{x \in S \mid x \leq a \text{ for some } a \in P\}$. Now, P_1 is an ideal of S and $L - P$ is a subsemilattice of S and $P_1 \cap (L - P) = \emptyset$. By the above theorem, there exists a prime ideal Q of S such that $P_1 \subset Q$ and $Q \cap (L - P) = \emptyset$. Now, clearly $Q \cap L = P$. The dual result follows from the above and from Lemma 1.7.

In general, the intersection of an arbitrary family of prime ideals of S need not be an ideal (see Corollary 2.14 below); however, we have the following which is dual to (viii) of Theorem 1.10.

Corollary 2.9. *Let I be an ideal of S . Then the intersection of all prime ideals of S containing I is again an ideal of S and is equal to I .*

Proof. If $a \in S$ and $a \notin I$, then $I \cap [a] = \emptyset$ and therefore by Theorem 2.6, there exists a prime ideal P of S such that $I \subset P$ and $P \cap [a] = \emptyset$ and hence $a \notin P$.

Corollary 2.10. *Every proper ideal (filter) of S is contained in a prime ideal (filter) of S and hence disjoint from a prime filter (ideal) of S .*

Corollary 2.11. *Every maximal ideal (filter) of S is prime.*

Corollary 2.12. *Let I be an ideal and $\{I_\alpha \mid \alpha \in \Delta\}$ a family of ideals of S . Then I is the smallest ideal containing every I_α , $\alpha \in \Delta$, if and only if, for any prime ideal P of S , $I \subset P$ is equivalent to $I_\alpha \subset P$ for every $\alpha \in \Delta$.*

Corollary 2.13. *If a and $b \in S$ and $a \neq b$, then there exists a prime ideal of S containing a and not containing b or vice-versa.*

Corollary 2.14. *$(S, \leq)$ is a lattice if and only if the intersection of an arbitrary family of prime ideals of S is an ideal of S .*

Proposition 2.15. *S has smallest (greatest) element if and only if the intersection of all prime ideals (filters) of S is non-empty.*

Proof. If a belongs to every prime ideal of S and if $x \in S$, then, by Corollary 2.9, $a \leq x$ and hence a is the smallest element of S . The converse is obvious, since the smallest element (if it exists) belongs to every prime ideal.

Proposition 2.16. *Suppose S has smallest element 0 and $A \subset S$.*

Then $\bigcap \{P \mid P \text{ is a prime ideal of } S \text{ not containing } A\} = \{x \in S \mid x \wedge a = 0 \text{ for all } a \in A\}$. Also, if S has greatest element 1 , then $\bigcap \{F \mid F \text{ is a prime filter of } S \text{ not containing } A\} = \{x \in S \mid x \vee a \text{ exists in } S \text{ and is equal to } 1 \text{ for all } a \in A\}$.

Proof. This follows from the definition of prime ideals (filters) and from the above proposition.

Definition 2.17. If S has smallest element 0 and L is a subsemi-lattice of S , then define $0(L) = \{x \in S \mid x \wedge y = 0 \text{ for some } y \in L\}$, i.e. $0(L) = \bigcup_{y \in L} (y)^*$.

By Corollary 1.12, it follows that $0(L)$ is an ideal of S .

Proposition 2.18. Suppose S has smallest element 0 and F is a proper filter of S . Then the following are equivalent:

- (i) F is a maximal filter.
- (ii) $S - F$ is a minimal prime ideal.
- (iii) $0(F) = S - F$.

Proof. The equivalence of (i) and (ii) follows from Lemma 1.7 and the equivalence of (ii) and (iii) is a direct verification.

Corollary 2.19. Suppose S has smallest element 0 and $d \in S$. Then $(d)^* = \{0\}$ if and only if d belongs to every maximal filter of S .

Corollary 2.20. Suppose S has smallest element 0 and greatest element 1 . Then the intersection of all maximal filters of S is $\{1\}$ if and only if 1 is the only element with the property that $(1)^* = \{0\}$.

§ 3. A class of congruences on a semilattice. For any subsemilattice L of a semilattice S , define

$$\theta_L = \{(x, y) \in S \times S \mid x \wedge a = y \wedge a \text{ for some } a \in L\}.$$

Then θ_L is a congruence on S and L is contained in a single equivalence class of θ_L . We denote the natural epimorphism of S onto the quotient semilattice S/θ_L by π_L .

Throughout this section, let S denote, unless otherwise stated, a distributive semilattice. If S is a distributive semilattice and F is a filter of S , then Katrinak ([10]) has proved that θ_F is the smallest congruence on S containing F in a single equivalence class and that S/θ_F is distributive. More generally, we have the following.

Theorem 3.1. *Let S be any semilattice which is directed above. Then the following are equivalent:*

- (i) S is distributive.
- (ii) S/θ_F is distributive for every subsemilattice L of S .
- (iii) S/θ_F is distributive for every filter F of S .

Proof. (i) $\Rightarrow$ (ii): Let S be distributive and L a subsemilattice of S . Let x, y and $z \in S$ such that $\pi_L(x) \wedge \pi_L(y) \leq \pi_L(z)$. This implies that there exists $a \in L$ such that $x \wedge y \wedge a = x \wedge y \wedge z \wedge a$ and therefore $(x \wedge a) \wedge (y \wedge a) \leq z$ so that there exist b and $c \in S$ such that $x \wedge a \leq b$, $y \wedge a \leq c$ and $b \wedge c = z$. Now, $\pi_L(x) = \pi_L(x \wedge a) \leq \pi_L(b)$, $\pi_L(y) = \pi_L(y \wedge a) \leq \pi_L(c)$ and $\pi_L(b) \wedge \pi_L(c) = \pi_L(z)$.

The implication (ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i): Let a, b and $c \in S$ such that $a \wedge b \leq c$. Choose $d \in S$ such that a, b and $c \in [d]$ and let $F = [d]$. Now, since S/θ_F is dis-

tributive, there exists e and $f \in S$ such that $\pi_F(a) \leq \pi_F(e)$, $\pi_F(b) \leq \pi_F(f)$ and $\pi_F(e) \wedge \pi_F(f) = \pi_F(c)$. Again there exist x, y and $z \in F$ such that $e \wedge f \wedge x = c \wedge x$, $e \wedge a \wedge y = a \wedge y$ and $f \wedge b \wedge z = b \wedge z$ which imply that $e \wedge f \wedge x = c$, $e \wedge a = a$ and $f \wedge b = b$ and therefore $a \leq e \wedge x$, $b \leq f \wedge x$ and $(e \wedge x) \wedge (f \wedge x) = c$. Hence S is distributive.

Note that the above theorem becomes trivial if S has largest element 1; for $\theta_{[1]}$ is the diagonal Δ_S of S . The following theorem and its proof are analogous to that of T. P. Speed [16].

Theorem 3.2. *Let S_1 and S_2 be two distributive semilattices and L_1 and L_2 subsemilattices of S_1 and S_2 respectively. If $f: S_1 \rightarrow S_2$ is a homomorphism such that $f(L_1) \subset L_2$, then there exists a unique homomorphism f_{12} such that*

$$\begin{array}{ccc} S_1 & \xrightarrow{\pi_{L_1}} & S_1/\theta_{L_1} \\ f \downarrow & & \downarrow f_{12} \\ S_2 & \xrightarrow{\pi_{L_2}} & S_2/\theta_{L_2} \end{array}$$

is commutative. Moreover, if f is an epimorphism, then f_{12} is also an epimorphism and if f is a monomorphism such that $f(L_1) = L_2$, then f_{12} is also a monomorphism.

Theorem 3.3. *Let L be a subsemilattice of S . If I' is an ideal of S/θ_L , then $I = \{x \in S \mid \pi_L(x) \in I'\}$ is an ideal of S . I' is a proper ideal of S/θ_L if and only if $I \cap L = \emptyset$. Also, if P is a prime ideal of S such that $P \cap L = \emptyset$, then $P' = \{\pi_L(x) \in S/\theta_L \mid x \in P\}$ is a prime ideal of S/θ_L . Further, $P \mapsto P'$ is an order-preserving bijection between the set of all prime ideals of S which are disjoint from L and the set of all prime ideals of S/θ_L . Moreover, the lattice of all filters of*

S/θ_L is isomorphic to the lattice of all filters of S containing L .

Proof. Let I' be an ideal of S and $I = \{x \in S \mid \pi_L(x) \in I'\}$. Clearly I is an initial segment of S . Also, if $\pi_L(a)$ and $\pi_L(b) \in I'$, then there exists $c \in S$ such that $\pi_L(c) \in I'$, $\pi_L(c) \leq \pi_L(b)$ and $\pi_L(b) \leq \pi_L(c)$ and there exist x and $y \in L$ such that $a \wedge c \wedge x = a \wedge x$ and $b \wedge c \wedge y = b \wedge y$ so that $a \wedge c \wedge x \wedge y = a \wedge x \wedge y$ and $b \wedge c \wedge x \wedge y = b \wedge x \wedge y$ and hence, by (vi) of Theorem 1.10, there exists $d \in S$ such that $a \leq d$, $b \leq d$ and $d \wedge c \wedge x \wedge y = d \wedge x \wedge y$. Now, $\pi_L(d) \leq \pi_L(c)$ and hence $d \in I$. Thus I' is an ideal. If $a \in I \cap L$, then, for any $x \in S$, $\pi_L(x) = \pi_L(x \wedge a)$ and hence $S/\theta_L = I'$. Conversely if $S/\theta_L = I'$, then for any $x \in L$, $\pi_L(x) \in I'$ and therefore $I \cap L \neq \emptyset$.

Now, it is a direct verification to prove that $I' \mapsto I$ induces an order-preserving bijection between the set of all prime ideals of S/θ_L and the set of all prime ideals of S which are disjoint from L and that $F \mapsto F' = \{\pi_L(x) \in S/\theta_L \mid x \in F\}$ is an isomorphism of the lattice of all filters of S containing L and the lattice of all filters of S/θ_L .

Theorem 3.4. Let L_1 and L_2 be two subsemilattices of S such that $L_1 \subset L_2$ and let $f_{12} : S/\theta_{L_1} \rightarrow S/\theta_{L_2}$ be the epimorphism defined by $f_{12}\pi_{L_1}(x) = \pi_{L_2}(x)$ for all $x \in S$. Then the following are equivalent:

- (i) f_{12} is an isomorphism.
- (ii) For any prime ideal P of S , $P \cap L_1 = \emptyset$ implies $P \cap L_2 = \emptyset$.
- (iii) The filter generated by L_2 is contained in the filter generated by L_1 .
- (iv) For any $a \in L_2$, there exists $b \in L_1$ such that $b \leq a$.

Proof. (i) $\Rightarrow$ (ii): If P is a prime ideal of S such that $P \cap L_1 =$

$\emptyset$, then, by the above theorem $\{\pi_{L_1}(x) \in S/\theta_{L_1} \mid x \in P\}$ is a prime ideal of S/θ_{L_1} and hence $\{\pi_{L_2}(x) \in S/\theta_{L_2} \mid x \in P\}$ is a prime ideal of S/θ_{L_2} , so that $P \cap L_2 = \emptyset$. The implication (ii) $\Rightarrow$ (iii) follows from Lemma 1.7 and (viii) of Theorem 1.10. The implication (iii) $\Rightarrow$ (iv) is obvious. Now assume (iv). It follows that $\theta_{L_2} \subset \theta_{L_1}$ and hence f_{12} is an isomorphism.

Theorem 3.5. *Let L be a subsemilattice of S and P' a prime ideal of S/θ_L . Let $P = \{x \in S \mid \pi_L(x) \in P'\}$. Then*

$$S/\theta_{S-P} = (S/\theta_L) / \theta_{S/\theta_L - P'}.$$

Proof. Since P and P' are prime ideals of S and S/θ_L respectively, $S-P$ and $S/\theta_L - P'$ are subsemilattices of S and S/θ_L respectively. Now, the theorem follows from Theorem 3.2 and from the following commutative diagrams

$$\begin{array}{ccc} S & \xrightarrow{\pi_{S-P}} & S/\theta_{S-P} \\ \downarrow & & \downarrow \\ S/\theta_L & \xrightarrow{\pi_{S/\theta_L - P'}} & (S/\theta_L) / \theta_{S/\theta_L - P'} \end{array}$$

where α is the map induced by the map $\pi_L: S \rightarrow S/\theta_L$ as in Theorem 3.2.

Theorem 3.6. *Let F be a filter of S . Then the following are equivalent:*

- (i) F is a prime filter.
- (ii) $\{\pi_F(x) \mid x \in S - F\}$ is an ideal of S/θ_F .
- (iii) S/θ_F has a unique maximal ideal.

Proof. The implication (i) $\Rightarrow$ (ii) follows from Lemma 1.7 and from Theorem 3.3.

(ii) $\Rightarrow$ (iii): We first observe that for any $a \in F$, $\pi_F(a)$ is the great-

est element of S/θ_F . Hence, every proper ideal of S/θ_F is contained in $\{\pi_F(x) \mid x \in S - F\}$ and therefore S/θ_F has a unique maximal ideal, viz., $\{\pi_F(x) \mid x \in S - F\}$.

(iii) $\Rightarrow$ (i): Let G be the unique maximal ideal of S/θ_F and let x and $y \in S - F$. Since the principal ideals of S/θ_F generated by $\pi_F(x)$ and $\pi_F(y)$ are proper, it follows that $\pi_F(x)$ and $\pi_F(y) \in G$ and therefore there exists $\pi_F(z) \in G$ such that $\pi_F(x) \leq \pi_F(z)$ and $\pi_F(y) \leq \pi_F(z)$, so that $x \wedge z \wedge a = x \wedge a$ and $y \wedge z \wedge a = y \wedge a$ for some $a \in F$. By Theorem 1.10, there exists $b \in S$ such that $x \leq b$, $y \leq b$ and $b \wedge z \wedge a = b \wedge a$ and hence $b \in S - F$; for otherwise $z \in F$ and hence $G = S/\theta_F$. Thus F is a prime ideal.

Theorem 3.7. *Let F be a filter of S . Then F is a maximal filter if and only if S/θ_F is the two element chain.*

Proof. Suppose F is a maximal filter. We know that for any x and $y \in F$, $\pi_F(x) = \pi_F(y)$ and that for any $x \in F$ and $y \in S$, $\pi_F(x) = \pi_F(y)$ implies $y \in F$. Now, let x and $y \in S - F$. Since F is maximal, there exist a and $b \in F$ such that $a \wedge x \leq y$ and $b \wedge y \leq x$ and hence $\pi_F(x) = \pi_F(a \wedge x) \leq \pi_F(y) = \pi_F(b \wedge y) \leq \pi_F(x)$, so that $\pi_F(x) = \pi_F(y)$. Conversely, suppose that S/θ_F is the two element chain. Suppose x and $y \in S - F$. Choose an element $z \in F$. Then, we have $\pi_F(z) \neq \pi_F(x)$ and hence $\pi_F(x) = \pi_F(y)$ so that there exists $a \in F$ such that $x \wedge a = y \wedge a$ and therefore $y \in F \vee [x]$. Hence, for any $x \notin F$, $F \vee [x] = S$. Thus F is maximal.

Now, we prove the following theorem which plays a vital role in proving a 'Grothendieck type' of representation theorem for distributive semilat-

tices by means of sections of sheaves [10], [11]. A part of this theorem was proved by Katrinak [7].

Theorem 3.8. *The map $F \mapsto \theta_F$ is an isomorphism from the lattice $F_0(S)$ of all non-empty filters of S into a permutable sublattice of the lattice $C(S)$ of all congruences on S . Hence $\{\theta_F \mid F \in F_0(S)\}$ is a distributive and permutable sublattice of $C(S)$. Moreover, for any family $\{F_\alpha \mid \alpha \in \Delta\} \subset F_0(S)$, θ_G , where $G = \bigvee_{\alpha \in \Delta} F_\alpha$, is the join of $\{\theta_F \mid \alpha \in \Delta\}$ in $C(S)$.*

Proof. Katrinak (3.3, [7]) has proved that, for any filters I and J of S , $\theta_I \vee \theta_J = \theta_{I \vee J} = \theta_I \circ \theta_J$ and now we prove that $\theta_I \cap \theta_J = \theta_{I \cap J}$. Clearly $\theta_{I \cap J} \subset \theta_I \cap \theta_J$. Let $(x, y) \in \theta_I \cap \theta_J$. There exist $a \in I$ and $b \in J$ such that $x \wedge a = y \wedge a$ and $x \wedge b = y \wedge b$. By Theorem 1.10, there exists $c \in S$ such that $a \leq c$, $b \leq c$ and $x \wedge c = y \wedge c$. Now, since $c \in I \cap J$, we have $(x, y) \in \theta_{I \cap J}$. Hence $F \mapsto \theta_F$ is a lattice homomorphism. Also, let I and $J \in F_0(S)$ and $\theta_I \subset \theta_J$. Let $x \in I$. Choose $y \in J$ and then $z \in S$ such that $x \leq z$ and $y \leq z$. Now, $(x, z) \in \theta_I \subset \theta_J$ and since $z \in J$, we have $x \in J$. Thus $I \subset J$. Hence $F \mapsto \theta_F$ is a lattice isomorphism of $F_0(S)$ onto the sublattice $\{\theta_F \mid F \in F_0(S)\}$ of $C(S)$. Finally let $\{F_\alpha \mid \alpha \in \Delta\} \subset F_0(S)$ and $F = \bigvee_{\alpha \in \Delta} F_\alpha$. Clearly $\bigvee_{\alpha \in \Delta} \theta_{F_\alpha} \subset \theta_F$. On the other hand, suppose $(x, y) \in \theta_F$ so that there exists $a \in F$ such that $x \wedge a = y \wedge a$. Now, $a \in F$ implies there exists $\alpha_1, \dots, \alpha_n \in \Delta$ such that $a \in G = \bigvee_{i=1}^n F_{\alpha_i}$ and hence $(x, y) \in \theta_G = \bigvee_{i=1}^n \theta_{F_{\alpha_i}} \subset \bigvee_{\alpha \in \Delta} \theta_{F_\alpha}$. This completes the proof of the theorem.

Remark. Note that, as in lattices, the lattice $C(S)$ of all congruences on a semilattice S need not be distributive, even when S is distributive; for, consider the following example: Let $I = \{1, 2, 3, \dots\}$ and $S =$

$\{0, a, b\} \cup I$ with the partial order given by $0 < a < n$, $0 < b < n$ for all $n \in I$ and $m < n$ whenever $m - n$ is positive for all m and $n \in I$. Then S becomes a distributive meet semilattice. Let

$$\theta = \{(a, b), (0, a), (0, b), (b, a), (a, 0), (b, 0)\} \cup \{(x, x) \mid x \in S\}$$

$$\alpha = \{(x, y) \in S \times S \mid x \wedge a = y \wedge a\} \quad \text{and} \quad \beta = \{(x, y) \in S \times S \mid x \wedge b = y \wedge b\}.$$

Then $(\theta \vee \alpha) \wedge \beta \neq (\theta \wedge \beta) \vee (\alpha \wedge \beta)$; for, $(1, b) \in \beta = (\theta \vee \alpha) \wedge \beta$ and $(1, b) \notin (\theta \wedge \beta) \vee (\alpha \wedge \beta)$.

Following Katrinak [10], a congruence θ on S is said to be $\vee$ -partial if $(a, b) \in \theta$, $(c, d) \in \theta$ imply that $(a \vee c, b \vee d) \in \theta$ whenever $a \vee c$ and $b \vee d$ exist in S . It can be easily seen that θ_L is $\vee$ -partial for any subsemilattice L of S . For any ideal I of S , let $\theta^I = \{(x, y) \in S \times S \mid [x] \cap [a] = [y] \cap [a] \text{ for some } a \in I\}$. Then θ^I is a $\vee$ -partial congruence on S . Also, for any $a \in S$, if $I = (a]$ and $F = [a]$, then $\theta^I \vee \theta_F = S \times S$ and $\theta^I \cap \theta_F = \Delta_S$, the diagonal of S . This observation is significant in the study of meet irreducible congruences on S ; for if S is a bounded distributive semilattice and θ is a meet irreducible congruence ((θ) is a prime ideal in $C(S)$), then, for any $a \in S$, $(0, a) \in \theta$ or $(a, 1) \in \theta$.

§ 4. Space of prime filters. In this section, we prove that every distributive semilattice is isomorphic to a subsemilattice of the semilattice of all open subsets of a topological space. As before, let S denote a distributive semilattice and X the set of all prime filters of S . For any $a \in S$, let $X_a = \{F \in X \mid a \in F\}$. Clearly, for any a and $b \in S$, $X_a \cap X_b = X_{a \wedge b}$ and if $a \vee b$ exists in S , then $X_a \cup X_b = X_{a \vee b}$. It follows that $\{X_a \mid a \in S\}$ forms a base for some topology $\mathcal{D}$ on X , which will be called the hull-kernel topology on X (The name hull-kernel topology is usually

used and is more reasonable when one considers the set Y of all prime ideals of S with the topology for which $\{Y_a \mid a \in S\}$ is a base, where $Y_a = \{P \in Y \mid a \notin P\}$, since in this topology, the closure of any $z \subset Y$ is $\{P \in Y \mid \bigcap z \subset P\}$. However, by Lemma 1.7, Y is homeomorphic with X). Also, for any $a \in S$, let $X'_a = \{F \in X \mid a \notin F\}$. It can be easily seen that $\{X'_a \mid a \in S\}$ also forms a base for some topology on X and we call this topology the dual hull-kernel topology and denote it by $\mathcal{D}'$. Now, the following theorem is a consequence of Theorem 1.10.

Theorem 4.1. *S is isomorphic to the meet subsemilattice $\{X_a \mid a \in S\}$ of the lattice of all open subsets of $(X, \mathcal{D})$.*

Theorem 4.2. *Let $a \in S$. Then the subspace X_a of $(X, \mathcal{D})$ is homeomorphic to the space $X[0, a]$ of all prime filters of $[0, a]$ with the hull-kernel topology.*

Proof. Using Corollary 2.8, it can be easily seen $F \mapsto F \cap [0, a]$ is a homeomorphism of X_a onto $X[0, a]$.

The following theorem and its proof are analogous to a similar theorem on distributive lattices proved by T. P. Speed [17].

Theorem 4.3. *Let S has smallest element 0 and Y the set of all maximal filters of S . Then Y is a Hausdorff and totally disconnected subspace of $(X, \mathcal{D})$. Moreover, the following are equivalent:*

- (i) $(Y, \mathcal{D})$ is compact.
- (ii) The restriction of $\mathcal{D}$ to Y is equal to the restriction of $\mathcal{D}'$ to Y .
- (iii) For each $a \in S$, there exist $a_1, a_2, \dots, a_n \in S$ such that $X'_a \cap Y$

$$= \bigcup_{i=1}^n (X_{a_i} \cap Y).$$

A semilattice S with 0 is said to be *pseudocomplemented* if, for each $a \in S$, there exists $a^* \in S$ such that $a \wedge b = 0$ if and only if $b \leq a^*$ for all $b \in S$. An element $a \in S$ is said to be *dense* if $a^* = 0$ and is said to be *closed* if $(a^*)^* = a$. It is well known that the set of closed elements of S forms a Boolean algebra under the partial ordering induced from that of S (see [3]).

Corollary 4.4. *Let S be a pseudocomplemented distributive semilattice. Then the space Y of all maximal filters of S with the hull-kernel topology is a Boolean space, i.e., a compact Hausdorff and totally disconnected space. In fact, Y is homeomorphic to the Stone space of the Boolean algebra of all closed elements of S .*

Proof. The first part follows from the above theorem and for the second part, $F \mapsto F \cap C(S)$ gives a homeomorphism of Y onto the Stone space of $C(S)$, the Boolean algebra of closed elements S .

The following proposition is a direct verification.

Proposition 4.5. *Let S be a distributive semilattice with 0 and X the space of prime filters with the hull-kernel topology. Then the following are equivalent:*

- (i) X is Hausdorff.
- (ii) X is a T_1 -space.
- (iii) Every prime filter of S is maximal.
- (iv) For every prime filter F of S , F is the only prime filter of

S such that $0(F) \cap F = \emptyset$.

Definition 4.6. An element $a \in S$ is said to be *semi-compact* if, for any $A \subset S$, $[a]$ is the smallest ideal containing A implies there exists a finite subset B of A such that $[a]$ is the smallest ideal containing B .

The following proposition follows from Corollary 2.12.

Proposition 4.7. Let $a \in S$. Then X_a is a compact subspace of $(X, \mathcal{D})$ iff a is semi-compact.

Proposition 4.8. If $(X, \mathcal{D})$ is compact, then S has largest element.

Proof. Suppose $(X, \mathcal{D})$ is compact. There exists $a_1, a_2, \dots, a_n \in S$ such that $X = \bigcup_{i=1}^n X_{a_i}$. Since S is directed above, there exists $a \in S$ such that $a_i \leq a$ for $1 \leq i \leq n$ so that a is the largest element of S .

Remark 4.9. The compactness of $(X, \mathcal{D})$ is still an open question, even if S has largest element 1. However, by the above two propositions, $(X, \mathcal{D})$ is compact if and only if S has largest element which is semi-compact. On the other hand, the space $(X, \mathcal{D}')$ is compact if and only if S has smallest element 0.

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