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Higher Order Boundary Value Problems for Differential Equations with Deviating Arguments

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This paper deals with n th order differential equations with deviating argument with suitable boundary conditions. The first result, Theorem 1 is an application of Schauder-Tychonoff fixed point theorem to establish the existence of a solution. In Theorem 2 contraction mapping principle is employed to determine an interval of uniqueness of the solution. An iterative scheme is provided in Theorem 3 which converges to the maximal solution, and it is used to compare the solutions in Theorem 4.

1. Introduction. In this paper, we are concerned with the following boundary value problem

$$(1) \quad x^{(n)}(t) + F(t, x(t), x(\phi_1(t)), x'(t), x'(\phi_2(t)), \dots, x^{(r)}(t), x^{(r)}(\phi_{r+1}(t))) = 0$$

defined on the interval $I = [a, b]$ with the boundary condition

$$(2) \quad \begin{cases} x^{(i)}(t) = \lambda^{(i)}(t), & i = 0, 1, \dots, n-2, \text{ for } t \leq a, \\ x^{(p)}(t) = \mu^{(p)}(t) & \text{for } t \geq b, \end{cases}$$

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where p is fixed and $0 \leq r \leq p \leq n-1$, $\phi_j(t)$ ($1 \leq j \leq r+1$) are given continuous functions defined on a suitable interval including I . The function $F: I \times \mathbb{R}^{2r+2} \rightarrow \mathbb{R}$ is assumed to be continuous. A solution of the boundary value problem (1), (2) is a continuously n times differentiable function satisfying (1) and (2) on the open interval (a, b) .

Let $E = I_a \cup I \cup I_b$, where $I_a = \{a\} \cup (\bigcup_{j=1}^{r+1} \phi_j(I) \cap (-\infty, a])$ and $I_b = \{b\} \cup (\bigcup_{j=1}^{r+1} \phi_j(I) \cap [b, +\infty))$, and let $B = C^{(k)}[I_a] \cap C^{(r)}[I] \cap C^{(p)}[I_b]$, where

$$k = \begin{cases} n-2, & \text{if } r \leq n-2 \\ n-1, & \text{otherwise.} \end{cases}$$

Define the norm $\| \cdot \|$ by $\|u\| = \max \{ |u^{(j)}(t)| : t \in E \text{ and } 0 \leq j \leq r \}$ for $u \in B$. Then $(B, \| \cdot \|)$ is a Banach space.

Let $\lambda \in C^{(k)}[I_a]$ and $\mu \in C^{(p)}[I_b]$. Consider a subset B' of B such that for $u \in B'$,

$$\begin{aligned} u^{(i)}(t) &= \lambda^{(i)}(t), \quad i = 0, 1, \dots, n-2, \quad t \in I_a, \text{ and} \\ u^{(p)}(t) &= \mu^{(p)}(t), \quad t \in I_b, \end{aligned}$$

i.e., u satisfies the boundary condition (2).

It can be easily seen that the Green's function associated with the boundary value problem

$$-x^{(n)}(t) = 0; \quad x^{(i)}(a) = 0, \quad i = 0, 1, \dots, n-2; \quad x^{(p)}(b) = 0$$

is given by

$$G(t, s) = \begin{cases} (t-a)^{n-1} \left(\frac{b-s}{b-a} \right)^{n-p-1} - (t-s)^{n-1}, & a \leq s \leq t \leq b, \\ (t-a)^{n-1} \left(\frac{b-s}{b-a} \right)^{n-p-1}, & a \leq t \leq s \leq b \end{cases}$$

and all its derivatives with respect to t up to order p are nonnegative.

Define

$$K(t, s) = \begin{cases} G(t, s) & \text{for } t \in I, \\ 0 & \text{for } t \notin I. \end{cases}$$

Then straightforward computations show that

$$\begin{aligned} \alpha_j &= \max_{a \leq t \leq b} \int_a^b \left| \frac{\partial^j K}{\partial t^j} \right| ds = \max_{a \leq t \leq b} \int_a^b \frac{\partial^j K}{\partial t^j} ds \\ &= \begin{cases} \frac{(n-j-1)^{n-j-1}}{(n-j)!} \left(\frac{b-a}{n-p} \right)^{n-j} & \text{if } n-1 > p = j, \\ b-a & \text{if } n-1 = p = j, \\ \frac{(p-j)(b-a)^{n-j}}{(n-p)(n-j)!} & \text{if } n-1 \geq p \geq j+1. \end{cases} \end{aligned}$$

Put

$$c(t) = \begin{cases} \lambda(t) & \text{if } t \leq a, \\ \sum_{r=0}^{n-2} \frac{\lambda^{(r)}(a)}{r!} (t-a)^r + [\mu^{(p)}(b) - \sum_{k=0}^{n-p-2} \frac{\lambda^{(p+k)}(a)}{k!} (b-a)^k] \times \\ \quad \frac{(n-p-1)!}{(n-1)!} \cdot \frac{(t-a)^{n-1}}{(b-a)^{n-p-1}} & \text{if } t \in I, \\ \mu(t) & \text{if } t \geq b. \end{cases}$$

2. Existence Theorem. We can prove the following result by a standard application of Schauder-Tychonoff fixed point theorem similar to that for $n = 2$ (see [4], [5]) and for $n = 3$ (see [1], [2]).

Theorem 1. Let M_j ($j = 0, 1, \dots, r$) be positive real numbers satisfying

$$\max_{t \in I_a} |\lambda^{(i)}(t)| \leq M_i, \max_{t \in I_b} |\mu^{(i)}(t)| \leq M_i, \text{ and } \max_{t \in I} |c^{(i)}(t)| \leq M_i$$

for every $i \in \{0, 1, \dots, r\}$, and let

$$Q = \sup \{ |F(t, x_0, y_0, \dots, x_r, y_r)| : t \in I, |x_i| \leq 2M_i \text{ and } |y_i| \leq 2M_i \text{ for } i = 0, 1, \dots, r \}.$$

Then the boundary value problem (1), (2) has a solution provided that

$$b - a \leq \begin{cases} (n-p) \left(\frac{(n-j)! M_j}{Q(n-j-1)^{n-j-1}} \right)^{1/(n-j)} & \text{if } n-1 > p = j, \\ \frac{M_j}{Q} & \text{if } n-1 = p = j, \\ \left(\frac{(n-j)! (n-p) M_j}{Q(p-j)} \right)^{1/(n-j)} & \text{if } n-1 \geq p \geq j+1, \end{cases}$$

for $j = 0, 1, \dots, r$.

3. Uniqueness Theorem. The following theorem follows from contraction mapping principle (for particular cases see [1], [2], [4], and [5]).

Theorem 2. Let K_j ($j = 0, 1, \dots, r$) be positive real numbers such that $\max_{t \in I_a} |\lambda^{(i)}(t)| \leq K_i$ and $\max_{t \in I_b} |\mu^{(i)}(t)| \leq K_i$ for every $i = 0, 1, \dots, r$. Let F satisfy a uniform Lipschitz condition

$$|F(t, x_0, y_0, \dots, x_r, y_r) - F(t, \bar{x}_0, \bar{y}_0, \dots, \bar{x}_r, \bar{y}_r)| \leq \sum_{i=0}^r [L_i |x_i - \bar{x}_i| + C_i |y_i - \bar{y}_i|]$$

on $D = \{(t, x_0, y_0, \dots, x_r, y_r) : t \in I, |x_i| \leq K_i \text{ and } |y_i| \leq K_i \text{ for every } i = 0, 1, \dots, r\}$. Then the boundary value problem (1), (2) has at most one solution $x(t)$ with $|x^{(j)}(t)| \leq K_j$ for every $j = 0, 1, \dots, r$ provided that $\sum_{i=0}^r (L_i + C_i) \max_{0 \leq i \leq r} \alpha_i < 1$.

4. Maximal Solution. Here we shall consider the case of $0 \leq r \leq n-2$. Let $B_c \subset B$ be the closed convex cone of functions having nonnegative derivatives up to order r and let $S \subset B_c$ be such that, for $u \in S$,

$$(3) \quad \begin{cases} u^{(i)}(t) = \alpha^{(i)}(t), & i = 0, 1, \dots, n-2; \quad t \in I_a \\ u^{(p)}(t) = \beta^{(p)}(t), & t \in I_b. \end{cases}$$

Define $S_\rho = \{x \in S : \|x\| \leq \rho\}$, and $\sigma(t)$ same as $c(t)$ changing $\lambda(t)$ to $\alpha(t)$ and $\mu(t)$ to $\beta(t)$.

Theorem 3. Suppose that the following conditions 1°-3° are satisfied:

1° $f(t, x_0, y_0, \dots, x_r, y_r)$ is continuous nonnegative and satisfies $f(t, \bar{x}_0, \bar{y}_0, \dots, \bar{x}_r, \bar{y}_r) \geq f(t, x_0, y_0, \dots, x_r, y_r)$ for all $(t, \bar{x}_0, \dots, \bar{y}_r)$, $(t, x_0, \dots, y_r) \in I \times \mathbb{R}_+^{2r+2}$ such that $\bar{x}_i \geq x_i$ and $\bar{y}_i \geq y_i$ for every $i = 0, 1, \dots, r$.

2° $\alpha^{(i)}(t) \geq 0$, $t \in I_a$; $\beta^{(i)}(t) \geq 0$, $t \in I_b$; $i = 0, 1, \dots, r$; $\alpha^{(j)}(a) \geq 0$, $j = 0, 1, \dots, n-2$; $\beta^{(p)}(b) \geq \sum_{k=0}^{n-p-2} \alpha^{(p+k)}(a)(b-a)^k/k!$.

3° There exists a $\rho > 0$ such that for all $t \in E$

$$\sigma^{(j)}(t) + \int_b^a \frac{\partial^j K}{\partial t^j} f(s, \rho, \dots, \rho) ds \leq \rho$$

for every $j = 0, 1, \dots, r$.

Define for each $t \in E$ the iterates

$$\begin{aligned} x_0(t) &= \int_b^a K(t, s) f(s, \rho, \dots, \rho) ds + \sigma(t), \\ x_n(t) &= \int_b^a K(t, s) f(s, x_{n-1}(s), x_{n-1}(\phi_1(s)), \dots, \\ &\quad x^{(r)}(s), x^{(r)}(\phi_{r+1}(s))) ds + \sigma(t) \end{aligned}$$

for each $n = 1, 2, \dots$. Then the sequence $\{x_n\}$ converges to the maximal solution of the equation

$$(4) \quad x^{(n)}(t) + f(t, x(t), x(\phi_1(t)), \dots, x^{(r)}(t), x^{(r)}(\phi_{r+1}(t))) = 0$$

satisfying the boundary condition (3) in S_ρ .

Proof. The fixed points of T defined by

$$(5) \quad Tx(t) = \sigma(t) + \int_b^a K(t, s) f(s, x(s), x(\phi_1(s)), \dots, x^{(r)}(s), x^{(r)}(\phi_{r+1}(s))) ds$$

for each $x \in S_\rho$ are solutions of the boundary value problem (3), (4). Now x_0 belongs to S_ρ , since $x_0^{(j)}(t) \leq \rho$ for every $j = 0, 1, \dots, r$. From 1°-3° we have

$$\begin{aligned}
x_1^{(j)}(t) &= \int_a^b \frac{\partial^j K}{\partial t^j} f(s, x_0(s), x_0(\phi_1(s)), \dots, x^{(r)}(\phi_{r+1}(s))) ds + \sigma^{(j)}(t) \\
&\leq \int_a^b \frac{\partial^j K}{\partial t^j} f(s, \rho, \dots, \rho) ds + \sigma^{(j)}(t) = x_0^{(j)}(t)
\end{aligned}$$

for each $j = 0, 1, \dots, r$, where we have used $x_0^{(j)}(\phi_{j+1}(t)) \leq \rho$ and which follows from 3°. Using an inductive argument it is easily seen that $\rho \geq x_0^{(j)}(t) \geq x_1^{(j)}(t) \geq \dots \geq x_n^{(j)}(t)$. Next, from the uniform continuity of $\partial^j K / \partial t^j$, $0 \leq j \leq r$, on $I \times I$, given $\varepsilon > 0$, there exists a $\delta_1(\varepsilon) > 0$ such that $|t_1 - t_2| < \delta_1(\varepsilon)$ implies

$$\|K(t_1, s) - K(t_2, s)\| < \frac{\varepsilon}{2(b-a) \sup_{t \in I} f(t, \rho, \dots, \rho)},$$

and for the same ε , there exists a $\delta_2(\varepsilon) > 0$ such that $\|\sigma(t_1) - \sigma(t_2)\| < \varepsilon/2$ whenever $|t_1 - t_2| < \delta_2(\varepsilon)$. Hence we have, for all $n = 0, 1, \dots$,

$$\begin{aligned}
|x_n^{(j)}(t_1) - x_n^{(j)}(t_2)| &\leq \int_a^b \left| \frac{\partial^j K(t_1, s)}{\partial t^j} - \frac{\partial^j K(t_2, s)}{\partial t^j} \right| f(s, \rho, \dots, \rho) ds \\
&\quad + |\sigma^{(j)}(t_1) - \sigma^{(j)}(t_2)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

whenever $|t_1 - t_2| < \min\{\delta_1(\varepsilon), \delta_2(\varepsilon)\}$ uniformly in n for all $j = 0, 1, \dots, r$. This shows that $\{x_n\}$ is equicontinuous. Therefore by Arzela's theorem $\{x_n\}$ is precompact and being monotonic, converges uniformly to some $x \in S_\rho$. From the continuity of f , x is a solution of the operator equation (5) or equivalently of (3), (4). Now let $y \in S_\rho$ be any other solution of (3), (4). Then

$$\begin{aligned}
x_0^{(j)}(t) - y^{(j)}(t) &= \int_a^b \frac{\partial^j K}{\partial t^j} [f(s, \rho, \dots, \rho) \\
&\quad - f(s, y(s), \dots, y^{(r)}(\phi_{r+1}(s)))] ds \\
&\geq 0.
\end{aligned}$$

In fact, induction shows that for each n , $x_n^{(j)}(t) - y^{(j)}(t) \geq 0$, $j = 0, 1, \dots, r$. Thus we have $x^{(j)}(t) - y^{(j)}(t) \geq 0$, proving the theorem.

5. Comparison Theorem.

Theorem 4. In addition to 1°-3° of Theorem 3, suppose that the following conditions are satisfied:

4° $|\lambda^{(i)}(t)| \leq \alpha^{(i)}(t)$ for $t \in I_a$ and $|\mu^{(i)}(t)| \leq \beta^{(i)}(t)$ for $t \in I_b$,
 $i = 0, 1, \dots, r$.

5° $|\lambda^{(i)}(a)| \leq \alpha^{(i)}(a)$, $i = 0, 1, \dots, n-2$ and

$$\begin{aligned} \left| \mu^{(p)}(b) - \sum_{k=0}^{n-p-2} \frac{\lambda^{(p+k)}(a)}{k!} (b-a)^k \right| \\ \leq \beta^{(p)}(b) - \sum_{k=0}^{n-p-2} \frac{\alpha^{(p+k)}(a)}{k!} (b-a)^k. \end{aligned}$$

6° $|F(t, x_0, y_0, \dots, x_r, y_r)| \leq f(t, |x_0|, |y_0|, \dots, |x_r|, |y_r|)$

for all $(t, x_0, y_0, \dots, x_r, y_r) \in I \times \mathbb{R}^{2r+2}$.

If $z \in S_p$ is any solution of the boundary value problem (1), (2), then $|z^{(j)}(t)| \leq x^{(j)}(t)$, $j = 0, 1, \dots, r$, for $t \in I$, where x is the maximal solution of (3), (4).

Proof. For $n = 1, 2, \dots$, set

$$y_n(t) = \int_a^b K(t, s) f(t, y_{n-1}(s), y_{n-1}(\phi_1(s)), \dots, y_{n-1}^{(r)}(s), y_{n-1}^{(r)}(\phi_{r+1}(s))) ds + \sigma(t)$$

with $y_0^{(j)}(t) = |z^{(j)}(t)|$. Then we have

$$\begin{aligned} y_1^{(j)}(t) &= \int_a^b \frac{\partial^j K}{\partial t^j} f(s, |z(s)|, |z(\phi_1(s))|, \dots, |z^{(r)}(s)|, \\ &\quad |z^{(r)}(\phi_{r+1}(s))|) ds + \sigma^{(j)}(t) \\ &\geq \int_a^b \frac{\partial^j K}{\partial t^j} |F(s, z(s), z(\phi_1(s)), \dots, z^{(r)}(s), z^{(r)}(\phi_{r+1}(s)))| ds \\ &\quad + |\sigma^{(j)}(t)| \\ &\geq |z^{(j)}(t)| \end{aligned}$$

on using the relations $|\sigma^{(j)}(t)| \geq |\sigma^{(j)}(t)|$, $j = 0, 1, \dots, r$. Inductively, it can readily be seen that

$$|z^{(j)}(t)| \leq y_1^{(j)}(t) \leq y_2^{(j)}(t) \leq \cdots \leq y_n^{(j)}(t) \leq \rho.$$

As in Theorem 3, $\{y_n\}$ is equicontinuous and, in view of the monotonicity, converges uniformly to a solution $y \in S_\rho$ of (3), (4). But the maximality of x completes the proof of Theorem 4.

Similar results to Theorems 3 and 4 have been obtained for a particular case $n = 2$, $p = 0$ in [3], [6]. Also in [6] F is taken as a positive function, whereas in our result it is not necessary.

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