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Sharma, Ashok K.

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A Generalization of Banach Contraction Principle
to 2-Metric Spaces

By Ashok K. SHARMA^{*}

The object of this paper is to generalize the well known Banach contraction principle to 2-metric spaces. For the related concepts of 2-metric space, see Gähler [2] and White [6].

Let (X, d) be a 2-metric space with 2-metric d and let f be a self mapping on X . Then f is said to be a *contraction* if there is an α such that $0 \leq \alpha < 1$ and

$$d(f(x), f(y), a) \leq \alpha d(x, y, a)$$

for all $x, y, a \in X$.

The analogue of Banach contraction principle in 2-metric spaces is the following:

Theorem A. *A contraction mapping f defined on a complete 2-metric space (X, d) has a unique fixed point u , and $f^n(x_0) \rightarrow u$ for any $x_0 \in X$.*

This principle has been generalized by various authors in many ways viz. Diminnie and White [1], Iséki [3], Sharma [4] and Rhoades [5].

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^{*} Department of Mathematics, G. D. Salwan College, Rajinder Nagar, Delhi-110060, India.

Definition 1. f is said to be (x_0, f) -orbitally continuous if $\lim_{i \rightarrow \infty} d(f^{n_i}(x_0), u, a) = 0$, for all $a \in X$ implies that $\lim_{i \rightarrow \infty} d(f(f^{n_i}(x_0)), f(u), a) = 0$ for all $a \in X$.

Definition 2 [4]. A 2-metric space (X, d) is said to be (x_0, f) -orbitally complete for some $x_0 \in X$, if every Cauchy sequence of the form $\{f^{n_i}(x_0)\}$ is convergent to some $u \in X$. This means

$$\lim_{i \rightarrow \infty} d(f^{n_i}(x_0), u, a) = 0, \text{ for all } a \in X.$$

In the present paper, we have considered a space X which is a 2-metric space with respect to two 2-metrics d_1 and d_2 . The following is our first result.

Theorem 1. Let X be a 2-metric space with respect to two 2-metrics d_1 and d_2 such that

- (1) $d_1(x, y, z) \leq d_2(x, y, z)$ for all $x, y, z \in X$,
- (2) (X, d_1) is complete,
- (3) $f: X \rightarrow X$ is continuous with respect to d_1 and
- (4) f is a contraction with respect to d_2 .

Then f has a unique fixed point $u \in X$ and $f^n(x_0) \rightarrow u$ for any $x_0 \in X$.

Proof. Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X , where $x_n = f(x_{n-1})$, $n \geq 1$. For any n and any $a \in X$ we have by (4)

$$\begin{aligned} d_2(x_n, x_{n+1}, a) &= d_2(f(x_{n-1}), f(x_n), a) \\ &\leq \alpha d_2(x_{n-1}, x_n, a), \end{aligned}$$

which implies that

$$d_2(x_n, x_{n+1}, a) \leq \alpha^n d_2(x_0, x_1, a).$$

Now, using this we observe that $d_2(x_0, x_1, x_k) = 0$ for all k . In fact,

let $d_2(x_0, x_1, x_k) = 0$ for some k . Then

$$\begin{aligned} d_2(x_0, x_1, x_{k+1}) &\leq d_2(x_0, x_1, x_k) + d_2(x_0, x_k, x_{k+1}) + d_2(x_k, x_1, x_{k+1}) \\ &\leq d_2(x_0, x_1, x_k) + \alpha^k d_2(x_0, x_0, x_1) + \alpha^k d_2(x_0, x_1, x_1) \\ &= d_2(x_0, x_1, x_k) = 0. \end{aligned}$$

Since $d_2(x_0, x_1, x_1) = 0$, the principle of finite induction yields that

$d_2(x_0, x_1, x_k) = 0$ for all k . On the other hand

$$d_2(x_n, x_{n+1}, x_m) \leq \alpha^n d_2(x_0, x_1, x_m),$$

and therefore $d_2(x_n, x_{n+1}, x_m) = 0$ for all m and n . Now, for any

$a \in X$ and $m > n$ we have

$$\begin{aligned} d_2(x_n, x_m, a) &\leq d_2(x_n, x_m, x_{n+1}) + d_2(x_n, x_{n+1}, a) + d_2(x_{n+1}, x_m, a) \\ &\leq \alpha^n d_2(x_0, x_1, a) + d_2(x_{n+1}, x_m, x_{n+2}) \\ &\quad + d_2(x_{n+1}, x_{n+2}, a) + d_2(x_{n+2}, x_m, a) \\ &\leq d_2(x_0, x_1, a) (\alpha^n + \alpha^{n+1}) + d_2(x_{n+2}, x_m, a) \\ &\leq \dots \\ &\leq d_2(x_0, x_1, a) \sum_{i=n}^{m-1} \alpha^i \\ &\rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$, since $\alpha < 1$. Thus, $\{x_n\}$ is a Cauchy sequence with respect to

d_2 and hence with respect to d_1 because of (1). Now by (2), there ex-

ists a point $u \in X$ such that $\lim_{n \rightarrow \infty} d_1(x_n, u, a) = 0$ for all $a \in X$ and on

using (3) we get $\lim_{n \rightarrow \infty} d_1(f(x_n), f(u), a) = 0$ for all $a \in X$. Now

$$\begin{aligned} d_1(u, f(u), a) &\leq d_1(u, f(u), x_{n+1}) + d_1(u, x_{n+1}, a) + d_1(x_{n+1}, f(u), a) \\ &\rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$. Thus, $d_1(u, f(u), a) = 0$ for all $a \in X$ and this yields $f(u)$

$= u$. If possible, let $f(v) = v$ for some $v \in X$ ($\neq u$). Then there exists

a point $z \in X$ with $d_2(u, v, z) \neq 0$ and therefore we have

$$0 < d_2(u, v, z) = d_2(f(u), f(v), z)$$

$$\leq \alpha d(u, v, z) < d(u, v, z),$$

a contradiction. Hence u is a unique fixed point of f .

The hypothesis of Theorem 1 can be weakened in the following ways.

Theorem 2. *If in Theorem 1, the conditions (2) and (3) are replaced by*

(5) *there exists a point $x_0 \in X$ such that the sequence $\{x_n\}$ as defined in Theorem 1 has a subsequence converging to some u in (X, d_1) and*

(6) *f is continuous at u with respect to d_1 ,*

even then u is a unique fixed point of f and $f^n(x_0) \rightarrow u$ for any $x_0 \in X$.

Theorem 3. *If in Theorem 1, the conditions (2) and (3) are replaced by*

(7) *f is (x_0, f) -orbitally continuous and*

(8) *X is (x_0, f) -orbitally complete, for some x_0 ,*

even then f has a unique fixed point u and $f^n(x) \rightarrow u$ for any $x \in X$.

Our last result is yet another generalization of Theorem A.

Remark. The conclusions of the Theorems 1, 2 and 3 still hold if we replaced the condition (4) by either of the following conditions:

$$(9) \quad d_2(f(x), f(y), a) \leq \alpha_1 d_2(x, f(x), a) + \alpha_2 d_2(y, f(y), a),$$

where $\alpha_i \in [0, 1)$ ($i = 1, 2$) and $\alpha_1 + \alpha_2 < 1$;

$$(10) \quad d_2(f(x), f(y), a) \leq \alpha_1 d_2(x, f(x), a) + \alpha_2 d_2(y, f(y), a) \\ + \alpha_3 d_2(x, y, a),$$

where $\alpha_i \in [0, 1)$ ($i = 1, 2, 3$) and $\alpha_1 + \alpha_2 + \alpha_3 < 1$;

$$(11) \quad d_2(f(x), f(y), a) \leq \alpha_1 d_2(x, f(x), a) + \alpha_2 d_2(y, f(y), a) \\ + \alpha_3 d_2(x, y, a) + \alpha_4 d_2(x, f(y), a) + \alpha_5 d_2(y, f(x), a),$$

where $\alpha_i \in [0, 1)$ ($i = 1, 2, \dots, 5$) and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$
 with $\alpha_1 = \alpha_2$ or $\alpha_3 = \alpha_4$.

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