Kobe University Repository : Kernel

PDF issue: 2026-11-22

Characters of Commutative Semigroups I

Anderson, A. James

(Citation)
Mathematics Seminar Notes,7(2):301-308

(Issue Date)
1979

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/E0001481

(URL)
https://hdl.handle.net/20.500.14094/E0001481

KOBE UNIVERSITY

MATHEMATICS SEMINAR NOTES
Vol. 7 (1979)

Characters of Commutative Semigroups I

By James A. ANDERSON

The purpose of the present paper is to give a description of the character semigroup for an arbitrary commutative semigroup. This problem has been solved in special cases by Hewitt and Zuckweman [3], Schwarz [8], and and Warne and Williams [9], and in the general case by Lesohin [4] and McAlister [6]. This description is a generalization of the description for semigroups with finite maximal semilattice structure found in Clifford and Preston and extends naturally to topological semigroups and bicharacters.

All semigroups in this paper are commutative. We shall also assume all semigroups have identity since this gives no loss of generality and eliminates the need for 0-characters and empty filters. All semilattices will be assumed to be lower semilattices unless stated otherwise. Q, R, C, and Z will denote the rational numbers, real numbers, complex numbers, and integers respectively.

Lemmas and theorems designated by * are generalizations of material in the last chapter of Clifford and Preston [1]. Proofs are given only when different from those given in Clifford and Preston [1].

The character of a semigroup is usually defined to be a mapping into either $R/Z \cup \{0\}$ or C. N. Kimura has suggested that for algebraic semi-

Received April 21, 1979.

groups, $Q/Z \cup \{0\}$ should be used instead. We shall use $Q/Z \cup \{0\}$ although all material concerning semigroup characters in this paper will be equally valid whichever definition is used.

Definition 1.1. Let S be a semigroup. A *character* of S is a mapping χ of S into $Q/Z \cup \{0\}$, not identically 0 such that $\chi(s + t) = \chi(s) + \chi(t)$ for all $s, t \in S$

Definition 1.2. Let S^* be the set of all characters of S. For χ, $\psi \in S^*$, define $\chi \circ \psi$ by $(\chi \circ \psi)(s) = \chi(s)\psi(s)$. $(S^*, \circ)$ is called the *character semigroup* of S.

Every union of groups G can be completely characterized by its greatest semilattice quotient A, the maximal subgroup G_α for each $\alpha \in A$ and connecting homomorphisms $g^\alpha_{\alpha'} : G_\alpha \to G_{\alpha'}$ for $\alpha > \alpha'$.

Definition 1.3. Call $(G_\alpha, g^\alpha_{\alpha'}, A)$ the *corresponding triple* of G, or simply *triple* of G. Denote the identity of G_α by 1_α.

Definition 1.4. Let A be a lower semilattice. A nonempty subset F of A is called a *filter* of A if the following conditions are satisfied:

(1) F is dual hereditary, i.e. if $a \in F$ and $a \leq x$ then $x \in F$.

(2) F is a subsemilattice.

Lemma 1.5.* *Let* $(G_\alpha, g^\alpha_{\alpha'}, A)$ *be the triple of a union of groups* S *and* G^0 *a group with* 0. *If* $f \in \mathrm{Hom}\,(S, G^0)$, *then* $\Delta_f = \{1_\alpha \mid f(1_\alpha) = 0\}$ *is a filter (Call* Δ_f *the filter associated with* f).

Notice in this lemma and in the remainder of this paper we shall use

the usual convention of identifying A with the semilattice of group identities.

Proof. Straightforward.

Lemma 1.6. *Let* F *be the set of filters of* A. *Let multiplication of filters be intersection. Then* $(\mathsf{F}, \cdot)$ *is a complete semilattice.*

Notation: Let S be a semigroup and G^0 a group with 0. Denote the semigroup Hom (S, G^0) by $S^{\#}$.

Lemma 1.7*. *Let* S *be a semigroup and* G^0 *a group with* 0. *Given* $\mu \in \mathsf{F}$, *let* $\overline{G}_\mu = \{f \mid f \in S^{\#}$ *and* $f(u) = 1$ *iff* $u \in \mu\}$. *Then* $\overline{G}$ *is a group with identity* ε_μ *defined by*

$$\varepsilon_\mu(g_\alpha) = \begin{cases} 1 & \text{if } \alpha \in \mu, \\ 0 & \text{otherwise.} \end{cases}$$

Also $\overline{G}_\mu \overline{G}_\nu \subset \overline{G}_{\mu \wedge \nu}$.

Theorem 1.8*. $S^{\#}$ *is the union of the semilattice* F *of groups* $\overline{G}_\mu$ $(\mu \in \mathsf{F})$ *where* $\overline{G}$ *is the group defined in Lemma* 1.7.

Definition 1.9. Let S be a commutative semigroup with triple $(G_\alpha, g^\alpha_{\alpha'}, A)$, and G^0 a group with 0. $f \in S^{\#}$ is called *principal* if there exists $\alpha \in A$ such that $f(1_\gamma) = 1$ iff $\gamma \geq \alpha$. α is called the *apex* of f.

Theorem 1.10*. *Let* S *be a commutative semigroup with triple* $(G_\alpha, g^\alpha_{\alpha'}, A)$, *and* G^0 *a group with* 0.

(a) *Let* $f \in S^{\#}$ *be principal, and* α *be the apex of* f. *Then the restriction* f' *of* f *to* G_α *belongs to* Hom (G_α, G^0), *and for every*

$s \in S$

$$f(s) = \begin{cases} f'(sl_\alpha) & \text{if } sl_\alpha \in G, \\ 0 & \text{otherwise.} \end{cases}$$

(b) *Let* $\alpha \in A$ *and* $f' \in \mathrm{Hom}\,(G_\alpha, G^0)$. *Then the function* f *above defines a principal function in* $S^\#$ *with apex* α, *and coinciding with* f' *on* G_α.

Theorem 1.11.* *Let* S *be a semigroup,* G^0 *a group with* 0, *and* S' *the maximal separative image of* S. *Then* $S^\# \cong S'^\#$.

Proof. Since G^0 is separative, for each $f \in S^\#$, there exists $f' \in S'^\#$ such that $f'\alpha = f$. Conversely given $f' \in S'^\#$, $f = f'\alpha$ belongs to $S^\#$. Since the homomorphic property of $f \to f'$ is preserved, $S^\# \cong S'^\#$.

The following lemma is due to T. Head [2] with the exception of (iv) which follows immediately from (iii).

Lemma 1.12. *Let* S *be a semigroup,* G^0 *a group with* 0, *and* U *an arbitrary union of groups.*

(i) $S \otimes Z$ *is a union of groups.*

(ii) *Let* $t : S \to S \otimes Z$ *be the canonical map. For every homomorphism* f $S \to U$, *there exists* $f' : S \otimes Z \to U$ *such that* $f = f't$.

(iii) $\mathrm{Hom}\,(S, U) \cong \mathrm{Hom}\,(S \otimes Z, U)$

(iv) $S^\# \cong (S \otimes Z)^\#$.

Let H and K be groups, and G^0 a group with 0. Let φ be a homomorphism of H into K. For each $f \in \mathrm{Hom}\,(K, G^0)$, define a mapping $\varphi^\#(f)$ of H into G^0 by $\varphi^\#(f(a)) = f(\varphi(a))$. Since $\varphi^\#(f) \in \mathrm{Hom}\,(H, G^0)$, the

mapping $\varphi^{\#}$ is a homomorphism of $K^{\#}$ into $H^{\#}$.

Definition 1.13. $\varphi^{\#}$ is called the G^0 *adjoint* of the homomorphism of H into K. If φ is a character, then $\varphi^{\#}$ is simply called the *adjoint* of φ and is denoted by φ^*.

Lemma 1.14.* *Let H and K be groups, and G^0 a divisible group with 0. If $\varphi \in \mathrm{Hom}\ (H, K)$ and $\varphi^{\#}$ is its adjoint, then $\varphi^{\#}(K^{\#}) \cong (\varphi(H))^{\#}$.*

Theorem 1.15.* *Let T be a union of groups with triple $(G_\alpha, g^\alpha_{\alpha'}, A)$ such that A is a lattice. Let G^0 be a group with 0. Let $A^{\#}$ be the dual of A and $\overline{T}$ the set of all principal functions in $T^{\#}$. Then $\overline{T}$ is a semigroup isomorphic with the union of the semilattice $A^{\#}$ of groups $G^{\#}_\alpha = \mathrm{Hom}\ (G_\alpha, G)$ with connecting homomorphisms $\theta^\alpha_\beta = (g^\beta_\alpha)^{\#}$, i.e. $\overline{T}$ has triple $(G^{\#}_\alpha, (g^\alpha_{\alpha'})^{\#}, A^{\#})$.*

Corollary 1. *Let T be a union of groups as above. Let A^* be the dual of A and $\overline{T}$ the set of all principal characters of T. Then $\overline{T}$ has triple $(G^*_\alpha, (g^\alpha_{\alpha'})^*, A^*)$.*

Corollary 2.* *If the conditions of Theorem 1.15 are satisfied, G^0 is divisible, and $\alpha, \beta \in A$ such that $\alpha \geq \beta$, then $\theta^\beta_\alpha(G_\beta)$ $(= G^{\#}_\beta \varepsilon_\alpha)$ is isomorphic with $(g^\alpha_\beta(G_\alpha))^{\#}$ $(= (G_\alpha 1_\beta)^{\#})$.*

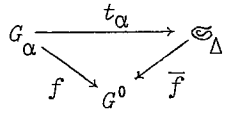
Corollary 3.* *If the conditions of Theorem 1.15 are satisfied and $\alpha, \beta \in A$ such that $\alpha \geq \beta$, then $\theta^\beta_\alpha(G_\beta) \cong g^\alpha_\beta(G_\alpha))^*$.*

Lemma 1.16. *Let T be a union of groups with triple $(G_\alpha, g^\alpha_{\alpha'}, A)$. Let F_V be the semilattice (upper) of filters of A. For $\Delta \in F$, let*

$\mathfrak{S}_\Delta = \varinjlim_{\Delta} \{G_\alpha \mid \alpha \in \Delta\}$ *with colimit cone* $\{t_\Delta^\alpha : G_\alpha \to \mathfrak{S}_\Delta\}$. *If* $\Delta \in \mathsf{F}$, *for* $\alpha \in \Delta$, t_Γ^α *is a homomorphism of* G_α *into* $\mathfrak{S}_\Gamma$. *This induces a homomorphism* $\varphi_\Gamma^\Delta : \mathfrak{S}_\Delta \to \mathfrak{S}_\Gamma$, *and a union of groups with triple* $(\mathfrak{S}_\Delta, \varphi_\Gamma^\Delta, \mathsf{F}_v)$ *is obtained* (*Notice* $\mathfrak{S}_\Delta$ *is also the greatest group image of the semigroup* $\bigcup_{\alpha\in\Delta} \mathfrak{S}_\alpha$).

Lemma 1.17. *Let* T *be a union of groups with triple* $(G_\alpha, g_{\alpha,}^\alpha, A)$ *and* $\mathfrak{S}$ *the union of groups with triple* $(\mathfrak{S}_\Delta, \varphi_\Gamma^\Delta, \mathsf{F}_v)$. *Then* $T^\#$ *is isomorphic to* $\mathfrak{S}$.

Proof. Let $f \in T^\#$ and Δ be the filter associated with f. By definition, there exists $\overline{f} : \mathfrak{S}_\Delta \to G$ such that



commutes for all $\alpha \in \Delta$. $\overline{f} \in \mathrm{Hom}\,(\mathfrak{S}_\Delta, G^0)$ and hence induces a principal homomorphism $\hat{f}$ on $\mathfrak{S}$. Conversely let $\omega \in \overline{\mathfrak{S}}$. There exists $\Delta \in \mathsf{F}$ which is the apex of ω. For $\alpha \in \Delta$, define $f_\alpha : G_\alpha \to G^0$ by $f_\alpha = t$. Define $f : T \to G^0$ by

$$f(g_\alpha) = \begin{cases} f_\alpha(g_\alpha), & \alpha \in \Delta, \\ 0, & \text{otherwise}. \end{cases}$$

Since $f \in T^\#$, it is easily seen that $f \to \hat{f}$ is an isomorphism.

Theorem 1.18. *Let* T *be a union of groups with triple* $(G_\alpha, g_{\alpha,}^\alpha, A)$ *and* $\mathfrak{S}$ *have triple* $(\mathfrak{S}_\Delta, \varphi_\Gamma^\Delta, \mathsf{F}_v)$ *as in Lemma* 1.16. *Let* $\mathsf{F}^\#$ *be the dual of* F_v. *Then* $T^\#$ *is a union of groups with triple* $(\mathfrak{S}_\Delta^\#, (\varphi_\Gamma^\Delta)^\#, \mathsf{F}^\#)$.

Proof. This follows directly from Lemma 1.16 and Theorem 1.14.

Corollary 1. *Let* T *and* $\mathfrak{S}$ *be as above, and* F^* *be the dual of* F_v.

Then T^* *is a union of groups with triple* $(\mathfrak{S}_\Delta^*, (\varphi_\Gamma^\Delta)^*, F^*)$.

Corollary 2. *Given the notation and hypothesis of Theorem* 1.17 *and* G^0 *divisible, if* Γ *and* $\Delta \in F$ *such that* $\Delta \leq \Gamma$, *then* $(\varphi_\Delta^\Gamma)^\#(\mathfrak{S}_\Delta^\#) \cong (\varphi_\Delta^\Gamma(\mathfrak{S}_\Gamma))^\#$.

Corollary 3. *Given the notation and hypothesis of Theorem* 1.17, *if* Γ *and* $\Delta \in F$ *such that* $\Delta \leq \Gamma$, *then* $(\varphi_\Delta^\Gamma) * (\mathfrak{S}_\Delta^*) \cong (\varphi_\Delta^\Gamma(\mathfrak{S}_\Gamma))^*$.

References

[1] A. H. Clifford and G. B. Preston: The algebraic theory of semigroups, Vol. 1, Amer. Math. Soc., Providence, R. I., 1961.

[2] T. J. Head: Homomorphisms of commutative semigroups as tensor maps, Jour. Nat. Sci. Math., 7 (1967), 155-159.

[3] E. Hewitt and H. S. Zuckerman: The ℓ_1-algebra of a commutative semigroup, Trans. Amer. Math. Soc., 83 (1956), 70-97.

[4] M. M. Lesohin: Characters of commutative semigroups I, Izv. Vyss. Učebn. Zaved. Mat., 1970, no. 8 (99), 67-74.

[5] M. M. Lesohin: Characters of commutative semigroups II, Izv. Vyss. Učebn. Zaved. Mat., 1971, no. 2 (105), 71-77.

[6] D. B. McAlister: Characters on commutative semigroups, Quart. Jour. Math. Oxford (2), 19 (1968), 141-157.

[7] M. Petrich: Some theorems concerning the dual of a commutative semigroup, Duke Math. Jour., 30 (1963), 25-32.

[8] S. Schwarz: Theory of characters of a finite commutative semigroup, Czech. Math. Jour., 4 (1954), 219-247.

[9] R. J. Warne and L. K. Williams: Characters on inverse semigroups, Czech. Math. Jour., 11 (1961), 150-155.

Department of Mathematics
North Arizona University
Flagstaff, Arizona 86001
U. S. A.