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(Citation)
Mathematics Seminar Notes,7(2):327-342

(Issue Date)
1979

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/E0001483

(URL)
https://hdl.handle.net/20.500.14094/E0001483

MATHEMATICS SEMINAR NOTES
Vol. 7 (1979)

On 2-Lattices over Real Quadratic Integers

By Yoshio MIMURA*

0. Introduction. Let F be a totally real number field and $\mathcal{O}$ be the ring of integers in F. Let V be a totally positive definite quadratic space over F with a symmetric bilinear form B and the associated quadratic form Q. A vector x in V is called a *q-vector* if $Q(x) = q \in F$. A lattice L, a finitely generated $\mathcal{O}$-module in V, is called a *q-lattice* if it is generated by some q-vectors over $\mathcal{O}$ and $B(L, L) \subset \mathcal{O}$. Note that any 1-lattice is isometric to a unit lattice, whose matrix is a unit matrix. Every 2-lattice is an even lattice. It is interesting to find all isometric classes of 2-lattices. Assume that F is the rational number field and $\mathbb{Z}$ is the ring of rational integers. In his paper [2], Kneser has found all isometric classes of 2-lattices over $\mathbb{Z}$ by using the theory of Lie algebras, and Ozeki [3] has found them after long and complicated calculations. In this note, we shall give them by a short and elementary argument. Furthere we shall give all classes of 2-lattices over $\mathcal{O}$ in the case of a real quadratic field F.

Most of the notations and terminology used here is taken from O'Meara [1].

Received May 11, 1979.

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1. Preliminaries. Let L be a lattice in a quadratic space over F, and L' be a lattice in another quadratic space over F. We say that L and L' are *isometric*, and write $L \cong L'$, if there is an isometry $\sigma: FL \to FL'$ such that $L\sigma = L'$ (see O'Meara [1, § 82A]). If there is a basis $\{x_1, \cdots, x_n\}$ over $\mathcal{O}$ for a lattice L, the discriminant dL of L is defined by $dL = \det(B(x_i, x_j))$, whose canonical image modulo squares of units is independent of the basis chosen for L. Clearly $dL = dM$ if $L \cong M$.

A quadratic space V over F is said to be *totally positive definite* if $V_{\mathfrak{p}}$ is positive definite for any infinite spot $\mathfrak{p}$ of F. A lattice L is called a *totally positive definite lattice* if so is FL.

Let L be a 2-lattice. Then $\mu(L)$, the set of 2-vectors in L, is finite since V is totally positive definite. A subset $\{x_1, \cdots, x_n\}$ of $\mu(L)$ is called a *minimal system of generators* in L if L is generated by $\{x_1, \cdots, x_n\}$ and L is not generated by any $n - 1$ 2-vectors. Clearly there exists a minimal system of generators $\{x_1, \cdots, x_n\}$ in L; we then write

$$L = [x_1, \cdots, x_n] \cong (B(x_i, x_j)),$$

where $(B(x_i, x_j))$ is an $n \times n$ matrix and called the *matrix* of L. If $\det(B(x_i, x_j)) \neq 0$ then the minimal system of generators is a basis for L.

Let L, L_1, L_2 be lattices. We write $L = L_1 \perp L_2$ if $L = L_1 + L_2$ and $B(L_1, L_2) = \{0\}$ (then $L_1 \cap L_2 = \{0\}$). L is said to be *decomposable* if there exist non-zero lattices L_1 and L_2 such that $L = L_1 \perp L_2$. If L is not decomposable it is said to be *indecomposable*. A vector x in L is said to be *reducible* if $x = y + z$ for some non-zero vectors $y, z \in L$

such that $B(y, z) = 0$.

Proposition. *Let L be a totally positive definite lattice. Then there exists an indecomposable splitting*

$$L = L_1 \perp \cdots \perp L_m.$$

The components $L_1, \cdots, L_m$ are unique (up to the ordering). Furthermore assume that L is a 2-lattice. Then each L_i is a 2-lattice and any 2-vector in L is irreducible.

Proof. The existence of the unique indecomposable splitting of L is ensured by [1; 105:1]. Assume that L is a 2-lattice. We have to show that any 2-vector in L is irreducible, which implies that L_1 is a 2-lattice. We can write $x = x_1 + \cdots + x_m$, $x_i \in L_i$. Then $2 = Q(x_1) + \cdots + Q(x_m)$. Since $Q(L)$ is contained in $2\mathcal{O}$, for each i we can find an $a_i \in \mathcal{O}$ such that $Q(x_i) = 2a_i$. Hence $1 = a_1 + \cdots + a_m$, and each a_i is a totally positive integer or zero. Assume that $a_1 \neq 0$ and $a_2 \neq 0$. Then $0 < a_1^{(j)} < 1$ for any conjugate of a_1. So we have $0 < \mathrm{Norm}(a_1) < 1$, which is a contradiction. Hence $x = x_i$ for some i.

The above proposition shows that we have only to classify indecomposable 2-lattices.

Remark 1. If a 2-lattice L contains a sublattice M $(\neq L)$ whose discriminant is a unit, then L is decomposable (see [1; 82:15]).

Remark 2. If $x_1, \cdots, x_n$ are in a 2-lattice L, then $Q(x_i)$ and $\det(B(x_i, x_j))$ are zero or totally positive.

Remark 3. Let $\{x_1, \cdots, x_n\}$ be a minimal system of generators in a

2-lattice L. If $Q(a_1x_1 + \cdots + a_nx_n) = 0$, then no a_i's are units (we have $a_1x_1 + \cdots + a_nx_n = 0$ since L is definite. If a_1 is a unit, then $x_1 = -a_1^{-1}(a_2x_2 + \cdots + a_nx_n) \in L$. Hence L is generated by $\{x_2, \cdots, x_n\}$, which contradicts the minimality of the set $\{x_1, \cdots, x_n\}$).

2. Case of the rational integers. In this section we consider 2-lattices over Z.

Lemma 1. *Any indecomposable 2-lattice has a minimal system of generators* $\{x_1, \cdots, x_n\}$ *such that* $B(x_1, x_i) = 1$ *for all* $i = 2, \cdots, n$.

Proof. Note that $B(u, v) \in \{0, 1, -1\}$ if $Q(u) = Q(v) = 2$ and $u \neq \pm v$ in a 2-lattice, since $0 < Q(u \pm v) = 4 \pm 2B(u, v)$. Take a minimal system of generators in the 2-lattice. We may assume that $B(x_1, x_i) = 1$ for $1 < i \leq s$ and $B(x_1, x_j) = 0$ for $j > s$. Since the lattice is indecomposable, s is greater than one and there exist i $(\leq s)$ and j $(> s)$ such that $B(x_i, x_j) \neq 0$. We can assume that $j = s + 1$ and $B(x_i, x_{s+1}) = 1$. Replace x_{s+1} by $x_i - x_{s+1}$. Then $B(x_1, x_k) = 1$ and $B(x_1, x_m) = 0$ for $1 < k \leq s + 1$ and $m > s + 1$. Repeating this procedure, we have the desired conclusion.

We shall concern with the following lattices:

$I_n = Ze_1 + Ze_2 + \cdots + Ze_n$, $B(e_i, e_j) = \delta_{ij}$, $n = 1, 2, \cdots$.

$A_n = \{x \in I_{n+1} \mid B(x, e_1 + \cdots + e_{n+1}) = 0\}$.

$B_n = \{x \in I_n \mid B(x, e_1 + \cdots + e_n) \equiv 0 \mod 2\}$.

$$C_n = \{x \in I_{n-1} + Z(e_n + \cdots + e_8) \mid B(x, e_1 + \cdots + e_8) \equiv 0 \mod 2\} + Z(e_1 + \cdots + e_8)/2, \text{ where } n \leq 8.$$

Lemma 2. *The lattices* A_n $(n \geq 1)$, B_n $(n \geq 3)$ *and* C_n $(8 \geq n \geq 4)$ *are* *indecomposable 2-lattices. Any two of them are not isometric except* $A_3 \cong B_3$, $A_4 \cong C_4$, *and* $B_5 \cong C_5$. *Moreover,*

	A_n	B_n	C_n
Discriminant	$n + 1$	4	$9 - n$
The number of 2-vectors	$n(n + 1)$	$2n(n - 1)$	$72, 126, 240$
Matrix	$\begin{pmatrix} 2 & & & \\ & 2 & & \\ & & \ddots & \\ & & & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & & \\ 0 & 2 & & \\ & & \ddots & \\ & & & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & 0 & & \\ 0 & 2 & & & \\ 0 & & 2 & & \\ & & & \ddots & \\ & & & & 2 \end{pmatrix}$

where the diagonal elements in the matrices are 2, *and any element which is not written is* 1.

Proof. We have

$$A_n = Z(e_1 - e_2) + \cdots + Z(e_1 - e_{n+1}) = [e_1 - e_2, \cdots, e_1 - e_{n+1}],$$

$$B_n = Z(e_1 + e_2) + Z(e_1 - e_2) + \cdots + Z(e_1 - e_n)$$
$$= [e_1 + e_2, e_1 - e_2, \cdots, e_1 - e_n], \text{ and}$$

$$C_n = Z(e_1 - e_2) + Z(e_1 + \cdots + e_8)/2 + Z(e_1 + e_2) + \cdots + Z(e_1 + e_{n-1})$$
$$= [e_1 - e_2, \tfrac{1}{2}(e_1 + e_2 + \cdots + e_8), e_1 + e_2, \cdots, e_1 + e_{n-1}].$$

So we have the above matrices (and these minimal systems are bases). Considering discriminants we may observe only three pairs $\{A_3, B_3\}$, $\{A_4, C_4\}$ and $\{B_5, C_5\}$. Indeed these pairs are those of isometric lattices: $A_3 = [e_1 - e_2, e_3 - e_4, e_1 - e_3] \cong B_3$, $A_4 = [e_1 - e_2, e_3 - e_5, e_4 - e_5, e_1 - e_5] \cong C_4$, and $B_5 = [e_2 + e_5, e_4 - e_1, e_3 - e_1, e_2 - e_1, e_5 - e_1] \cong C_5$.

Lemma 3. *Let* L *be an indecomposable 2-lattice with a minimal system of generators* $\{x, x_1, \cdots, x_n\}$. *Assume that* $B(x_i, x_j) = 1$ *whenever* $i \neq j$. *Then there exists a minimal system of generators* $\{x', x_1, \cdots, x_n\}$

such that $B(x', x_i) \in \{0, 1\}$, $0 \le s \le 2$ *and* $2s < n$, *where* s *denotes the number of* i*'s with* $B(x', x_i) = 0$. *Further* $n \le 7$ *if* $s = 2$.

Proof. Since L is indecomposable, $B(x, x_i) = \pm 1$ for some i. Letting $x'' = B(x, x_i)x$ we have $B(x'', x_i) = 1$. If $B(x'', x_j) = -1$ then we have $Q(x'' + x_j - x_i) = 0$, a contradiction (cf. Remark 3). So $B(x'', x_j) \in \{0, 1\}$. Letting $x' = x''$ if $2s < n$ and letting $x' = x_i - x''$ if $2s \ge n$, we have $2s < n$. If $s \ge 3$ (we assume $B(x', x_j) = 0$ for $1 \le j \le 3$ and $B(x', x_j) = 1$ for $4 \le j \le 7$) then $Q(2x' + x_1 + x_2 + x_3 - x_4 - x_5 - x_6 - x_7) = 0$. This is a contradiction by Remark 3. Thus $s \le 2$. Assume $s = 2$. Then we have $Q(3x' + 2x_1 + 2x_2 - x_3 - \cdots - x_8) = 0$ if $n \ge 8$ (and we assume $B(x', x_1) = B(x', x_2) = 0$ and $B(x', x_3) = \cdots = B(x', x_8) = 1$). This is a contradiction by Remark 3.

Theorem A. *An indecomposable 2-lattice over* $\mathbb{Z}$ *is isometric to one of the lattices* A_n $(n \ge 1)$, B_n $(n \ge 4)$, *and* C_n $(n = 6, 7, 8)$.

Proof. We shall prove the theorem by induction on the cardinal number of a minimal system of generators of an indecomposable 2-lattice L. If the number is one or two then $L \cong A_1$ or $L \cong A_2$ trivially. Let $\{y, z, x_1, \cdots, x_n\}$ be a minimal system of generators in L, where $n \ge 1$. By Lemma 1 we may assume that $B(x_n, x_i) = B(x_n, z) = B(x_n, y) = 1$ for $1 \le i < n$. Since the lattice Z generated by $\{z, x_1, \cdots, x_n\}$, we can assume that $Z \cong A_{n+1}$, $Z \cong B_{n+1}$ or $Z \cong C_{n+1}$, and that $B(x_i, x_j) = 1$ if $i \ne j$ and $B(z, x_k) = 0$ or 1 according to $k \le s$ or $k > s$ by Lemma 3. Let M be the lattice generated by $\{x_1, \cdots, x_n\}$. If $B(y, x_i) = 0$ for $1 \le i \le n$, then $B(y, z) = \pm 1$ because L is indecomposable. Replacing y by $z \mp y$, we can assume that $Y = M + \mathbb{Z}y$ is indecomposable.

By Lemma 3 we can suppose that the matrix of $y, z, x_1, \cdots, x_n$ is

$$\left(\begin{array}{cc|ccc} 2 & a & & F & \\ a & 2 & & & \\ \hline & & 2 & & \\ & {}^tF & & \ddots & \\ & & & & 2 \end{array}\right)$$

where $a \in \{0, 1, -1\}$, F is a $2 \times n$ matrix with rows such that at most two elements are 0 and the other elements are 1, and the (i, j)-element for i, $j \geq 3$ is 1 or 2 according as $i \neq j$ or $i = j$. Thus we have only to treat the following nine cases, and hence $a \neq -1$ (if $a = -1$ then we have $Q(y + z - x_n) = 0$, a contradiction). Put

$$F_1 = \begin{pmatrix} 1 & 0 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \end{pmatrix},\ n \geq 3; \qquad F_2 = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \end{pmatrix},\ n \geq 3;$$

$$F_3 = \begin{pmatrix} 1 & 0 & 0 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 \end{pmatrix},\ 7 \geq n \geq 5; \quad F_4 = \begin{pmatrix} 0 & 0 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \end{pmatrix},\ 7 \geq n \geq 5;$$

$$F_5 = \begin{pmatrix} 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \end{pmatrix},\ 7 \geq n \geq 5; \quad F_6 = \begin{pmatrix} 1 & 0 & 0 & 1 & \cdots & 1 \\ 0 & 1 & 0 & 1 & \cdots & 1 \end{pmatrix},\ 7 \geq n \geq 5;$$

$$F_7 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & \cdots & 1 \\ 0 & 1 & 0 & 1 & 1 & \cdots & 1 \end{pmatrix},\ 7 \geq n \geq 5.$$

We have $n = 5, 6$ in the case that $F \in \{F_3, F_4, F_5, F_6, F_7\}$ by Remark 1 and Lemma 2 since $dC_8 = 1$.

(1) If F has a row whose elements are equal to 1 then $L \cong A_{n+2}$, $L \cong B_{n+2}$ or $L \cong C_{n+2}$ by Lemma 3.

(2) If $F \in \{F_1, F_3, F_6, F_7\}$ with $a = 1$, then there are four vectors, say u_1 $(= y)$, u_2 $(= z)$, u_3, u_4 in $\{y, z, x_1, \cdots, x_n\}$, whose matrix is

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix},$$

and hence we have $Q(u_1 - u_2 - u_3 + u_4) = 0$, which is impossible.

(3) If $F \in \{F_2, F_4, F_5, F_6\}$ with $a = 0$, then there are five vectors, say u_1 $(= y)$, u_2 $(= z)$, u_3, u_4, u_5 in $\{y, z, x_1, \cdots, x_n\}$, whose matrix is

$$\begin{pmatrix} 2 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix},$$

and hence we have $Q(u_1 + u_2 + u_3 - u_4 - u_5) = 0$, which is impossible.

(4) If $F = F_5$ with $a = 1$, then we have $Q(y + z + x_1 + x_2 - x_3 - x_4 - x_5) = 0$, which is impossible.

(5) If $F = F_7$ with $a = 0$, then $Q(y - z - x_1 + x_2 - x_3 + x_4) = 0$, which is impossible.

(6) If $F = F_1$ with $a = 0$, then $L = [x_1, y - x_1 + x_2, z, x_2, \cdots, x_n] \cong C_{n+2}$.

(7) If $F = F_4$ with $a = 0$, then $L = [x_3, z + x_1 - x_3, y - x_1 + x_2, z, x_2, x_4, \cdots, x_n] \cong C_{n+2}$.

(8) If $F = F_4$ with $a = 1$, then $L = [z, y - z + x_2, x_1, \cdots, x_n] \cong C_{n+2}$.

(9) If $F = F_2$ with $a = 1$, then $L \cong C_{n+2}$ trivially.

3. Case of real quadratic integers. Let $\mathbf{Q}$ denote the rational number field, and a be in a real quadratic field F. The conjugate of a will be denoted by $\bar{a}$. Put $\zeta = \frac{1}{2}(1 + \sqrt{5})$

Lemma 4. *Let $\{x_1, x_2\}$ be a minimal system of generators in a 2-lattice. Then $B(x_1, x_2) \in \{0, \pm 1, \pm\sqrt{2}, \pm\sqrt{3}, \pm\zeta, \pm\bar{\zeta}\}$.*

Proof. Put $a = B(x_1, x_2)$. If $a = \pm 2$ then $Q(x_1 \mp x_2) = 0$, which is a contradiction by Remark 3. Hence $-2 < \alpha < 2$ and $-2 < \bar{\alpha} < 2$ by Remark 2. Therefore $\alpha + \bar{\alpha} \in \{0, \pm 1, \pm 2, \pm 3\}$ and $\alpha\bar{\alpha} \in \{0, \pm 1, \pm 2, \pm 3\}$. This implies that α is in $\{0, \pm 1, \pm\sqrt{2}, \pm\sqrt{3}, \pm\zeta, \pm\bar{\zeta}\}$.

Let $e_1, \cdots, e_n$ be an orthonormal system of vectors in V, i.e., $B(e_i, e_j) = \delta_{ij}$ for all i, j. We shall concern with the following lattices, where $e = e_1 + \cdots + e_n$.

$I_n = \mathcal{O}e_1 + \cdots + \mathcal{O}e_n$.

$A_{n-1} = \{x \in I_n : B(x, e) = 0\} = [e_1 - e_2, \cdots, e_1 - e_n]$, $n \geq 2$.

$B_n = \{x \in I_n : B(x, e) \equiv 0 \bmod 2\} = [e_1 + e_2, e_1 - e_2, \cdots, e_1 - e_n]$, $n \geq 2$.

$C_n = \{x \in I_{n-1} + \mathcal{O}(e_n + \cdots + e_8) : B(x, e') \equiv 0 \bmod 2\} + \frac{1}{2}\mathcal{O}e'$

$= [e_1 - e_2, \frac{1}{2}e', e_1 + e_2, \cdots, e_1 + e_{n-1}]$, $4 \leq n \leq 8$, where $e' = e_1 + \cdots + e_8$.

$D_n = \{x \in I_n : B(x, e) \equiv 0 \bmod \sqrt{2}\}$

$= [\sqrt{2}e_1, e_1 + e_2, \cdots, e_1 + e_n]$, $n \geq 2$, where $F = \mathbb{Q}(\sqrt{2})$.

$D'_4 = D_4 + \mathcal{O}(e_1 + e_2 + e_3 + e_4)/\sqrt{2}$

$= [\sqrt{2}e_1, (e_1 + e_2 + e_3 + e_4)/\sqrt{2}, e_1 + e_3, e_1 + e_4]$.

$T_2 = [e_1 - e_2, \frac{1}{2}(1 + \sqrt{3})e_1 + \frac{1}{2}(1 - \sqrt{3})e_2]$, where $F = \mathbb{Q}(\sqrt{3})$.

$F_n = [\frac{1}{2}(1 - \zeta)(-\zeta^3 e_1 + e_2 + e_3 + e_4), e_1 - e_2, \cdots, e_1 - e_n]$, $2 \leq n \leq 4$, where $F = \mathbb{Q}(\sqrt{5})$.

Remark 4. The lattice F_n has another basis. In fact, if $\{x_0, x_1, \cdots, x_{n-1}\}$ is the basis given above, then we have

$F_3 = [\zeta^2 x_0 - \zeta x_1 - \zeta x_2, \zeta^2 x_0 - x_1 - \zeta x_2, \zeta^2 x_0 - \zeta x_1 - x_2]$ and

$F_4 = [x, x - \bar{\zeta}x_1, x - \bar{\zeta}x_2, x - \bar{\zeta}x_3]$, where $x = -\zeta^4 x_0 + \zeta^2(x_1 + x_2 + x_3)$.

Lemma 5. *The lattices* A_n $(n \geq 1)$, B_n $(n \geq 3)$, C_n $(8 \geq n \geq 4)$, D_n $(n \geq 2)$, D'_4, F_n $(4 \geq n \geq 2)$ *and* T_2 *are indecomposable* 2-*lattices. Any tow of* *them are not isometric except* $A_3 \cong B_3$, $A_4 \cong C_4$, *and* $B_5 \cong C_5$. *Moreover*

	A_n	B_n	C_n
Discriminant	$n + 1$	4	$9 - n$
The number of 2-vectors	$n(n + 1)$	$2n(n - 1)$	72,126,240
Matrix	$\begin{pmatrix} 2 & & & \\ & 2 & & \\ & & \ddots & \\ & & & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & & \\ 0 & 2 & & \\ & & \ddots & \\ & & & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & 0 & & \\ 0 & 2 & & & \\ 0 & & 2 & & \\ & & & \ddots & \\ & & & & 2 \end{pmatrix}$

D_n	D'_4	F_n	T_2
2	1	$3 - \zeta$, 2, 1	1
$2n^2$	48	10, 30, 120	12
$\begin{pmatrix} 2 & \sqrt{2} & \cdots & \sqrt{2} \\ \sqrt{2} & 2 & & \\ \vdots & & \ddots & \\ \sqrt{2} & & & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 & \sqrt{2} & \sqrt{2} \\ 1 & 2 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 2 & 1 \\ \sqrt{2} & \sqrt{2} & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & \zeta & \cdots & \zeta \\ \zeta & 2 & & \\ \vdots & & \ddots & \\ \zeta & & & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix}$

where the diagonal elements in the matrices are 2, *the first row and the first column of* D_n *consist of several* $\sqrt{2}$*'s and one* 2, *those of* F_n *consist of several* ζ*'s and one* 2, *and the other elements which are not specified are* 1.

Proof. The above basis of each lattice gives the corresponding matrix. Each lattice is totally positive definite since so is I_n. The matrices show that the lattices are indecomposable by Proposition in section 1.

Lemma 6. *Let* L *be an indecomposable 2-lattice with a minimal system of generators* $\{x, x_1, \cdots, x_n\}$. *Assume that* $B(x_i, x_j) = 1$ *whenever* $i \neq j$. *Then there exists a minimal system of generators* $\{x', x_1, \cdots, x_n\}$ *such that the vector* $Y' = (B(x', x_1), \cdots, B(x', x_n))$ *is one of the fol-*

lowing:

$(1, \cdots, 1)$ *with* $n \geq 1$, *i.e.*, $B(x', x_i) = 1$ *for all* i.

$(1, \cdots, 0, \cdots, 1)$ *with* $n \geq 3$, *i.e.*, $B(x', x_i) = 0$ *for exactly one* i *and* $B(x', x_j) = 1$ *for all* $j \neq i$.

$(1, \cdots, 0, \cdots, 0, \cdots, 1)$ *with* $7 \geq n \geq 5$, *i.e.*, $B(x', x_i) = 0$ *for exactly two* i*'s and* $B(x', x_j) = 1$ *for other* j.

$(\sqrt{2}, \cdots, \sqrt{2})$ *with* $n \geq 1$, *i.e.*, $B(x', x_i) = \sqrt{2}$ *for all* i.

$(\zeta, \cdots, \zeta)$ *with* $3 \geq n \geq 1$, *i.e.*, $B(x', x_i) = \zeta$ *for all* i.

$(\sqrt{3})$ *with* $n = 1$.

Proof. We proceed by induction on n. Put

$$Y = (B(x, x_1), \cdots, B(x, x_n)).$$

For $n = 1$, after replacing x by $-x$ if necessary, we have $B(x, x\ ) \in \{1, \sqrt{2}, \sqrt{3}, \zeta, \bar{\zeta}\}$ by Lemma 4, and $L = [x, \bar{\zeta}(x_1 - x)] \cong \begin{pmatrix} 2 & \zeta \\ \zeta & 2 \end{pmatrix}$ if $B(x, x_1) = \bar{\zeta}$. Next let $n \geq 2$. Then $\{x, x_2, \cdots, x_n\}$ is a minimal system of generators in $[x, x_2, \cdots, x_n]$. We can assume that $[x, x_2, \cdots, x_n]$ is indecomposable by replacing x by $ax_1 - x$ if necessary. Hence by the hypothesis of induction there is a minimal system $\{x', x_1, \cdots, x_n\}$ such that $Y' = (a, X)$, X being one of the vectors listed in the lemma. Put

$$D = \det \left(\begin{array}{c|cccc} 2 & & Y' & & \\ \hline & 2 & & & \\ {}^tY & & \ddots & & \\ & & & & 2 \end{array}\right) = (a_1 + \cdots + a_n)^2 - (n+1)(a_1^2 + \cdots + a_n^2 - 2),$$

where $Y' = (a_1, \cdots, a_n)$. Thus D is zero or totally positive by Remark 2. If $a = \pm\sqrt{3}$ then L contains $[x', x_1]$, and the discriminant of $[x', x_1]$ is one. This contradicts the indecomposability of L by Remark 1.

(1) If $a = 0$ and $a = 2$ (and $X = (b)$ with $b \in \{1, \sqrt{2}, \zeta\}$), then we obtain $Y' = (b, b)$ by replacing x' with $bx_2 - x'$.

(2) $X = (1, \cdots, 1)$. If $a = 0$ with $n \geq 3$ or $a = 1$ with $n \geq 1$, then we obtain Y' as required. If $a \notin \{0, 1\}$ then $a \in \{-1, \zeta, \bar{\zeta}\}$ with $n = 2$ or $a \in \{\zeta, \bar{\zeta}\}$ with $n = 3$ since $D = (2 - a)(2 + an)$. If $a = -1$ and $n = 2$ then we have $Q(x' + x_1 - x_2) = 0$, a contradiction by Remark 3. If $a = \zeta$ (resp. $\bar{\zeta}$) with $n = 2$, then replacing x' by $x_1 - \bar{\zeta}x_2 + \bar{\zeta}x'$ (resp. $x_1 - \bar{\zeta}x'$), we have $Y' = (\zeta, \zeta)$. If $a = \zeta$ (resp. $\bar{\zeta}$) with $n = 3$, then replacing x' by $\bar{\zeta}^2(-\bar{\zeta}x_1 + x_2 + x_3 - \bar{\zeta}x')$ (resp. $x_1 - \bar{\zeta}x'$), we have $Y' = (\zeta, \zeta, \zeta)$.

(3) $X = (0, 1, \cdots, 1)$ with $n \geq 4$. If $a = 1$ then we obtain $Y' = (0, 1, \cdots, 1)$ as required. If $a = 0$ and $n = 4$, then replacing x' by $x_4 - x'$, we have $Y' = (0, 1, 1, 1)$. If $a = 0$ and $n \geq 5$, then $Y' = (0, 0, 1, \cdots, 1)$, and $n \leq 7$ by Remark 1 since $d[x', x_1, \cdots, x_7] = 1$. If $a \notin \{0, 1\}$ then $a \in \{\zeta, \bar{\zeta}\}$ and $n = 4$, since $D = 8 - 4a - n(a - 1)^2$. If $a = \zeta$ (resp. $\bar{\zeta}$) then we have $Q((2 + \zeta)x - 2\zeta x_1 + \zeta^2 x_2 - x_3 - x_4) = 0$ (resp. $Q((2 + \bar{\zeta})x - 2\bar{\zeta}x_1 + \bar{\zeta}^2 x_2 - x_3 - x_4) = 0$), a contradiction by Remark 3.

(4) $X = (0, 0, 1, \cdots, 1)$ with $8 \geq n \geq 6$. We may assume $n \leq 7$ by Remark 1 since $d[x', x_2, \cdots, x_8] = 1$. We have $a = 0$ or $a = 1$ since $D = 14 - 6a - n(a^2 - 2a + 2)$. If $a = 1$ then we obtain Y' as required. If $a = 0$ with $n = 6$, then replacing x' by $x_6 - x'$, we have $Y' = (0, 0, 1, 1, 1, 1)$. If $a = 0$ with $n = 7$, then we have

$$Q(2x + x_1 + x_2 + x_3 - x_4 - x_5 - x_6 - x_7) = 0,$$

a contradiction by Remark 3.

(5) $X = (\sqrt{2}, \cdots, \sqrt{2})$. If $a = 2$, then we have Y' as required. If

$a \neq \sqrt{2}$ then we have $a = 0$ with $n \in \{2, 3\}$ since $D = 6 - 2n - na^2 + 2a(n - 1)\sqrt{2}$. But $Q(\sqrt{2}x' + x_1 - x_2 - x_3) = 0$ if $a = 0$ and $n = 3$. This is a contradiction.

(6) $X = (\zeta)$ with $n = 2$. We have $a \in \{0, 1, \zeta, -\bar{\zeta}\}$ since $D = 6 - 2a^2 - 2\zeta^2 + 2a\zeta$. If $a = 1$ (resp. $\zeta, -\bar{\zeta}$) then replacing x' by $-\bar{\zeta}x_1 + x_2 + \bar{\zeta}x'$ (resp. x', $-\bar{\zeta}(x_1 + x')$), we have $Y' = (\zeta, \zeta)$.

(7) $X = (\zeta, \zeta)$ with $n = 3$. We have $a \in \{1, \zeta, -\bar{\zeta}\}$ since $D = 8 - 3a^2 + 4a\zeta - 4\zeta^2$. If $a = 1$ (resp. $\zeta, -\bar{\zeta}$) then replacing x' by $(\zeta - 2)(x' - x_1) - \bar{\zeta}(x_2 + x_3)$ (resp. x', $-\bar{\zeta}(x' + x_1)$), we have $Y' = (\zeta, \zeta, \zeta)$.

(8) $X = (\zeta, \zeta, \zeta)$ with $n = 4$. This case does not occur by Remark 1 since $d[x', x_2, x_3, x_4] = 1$.

(9) $X = (\sqrt{3})$ with $n = 2$. This does not occur by Remark 1 since $d[x', x_2] = 1$.

Theorem B. *Let F be a real quadratic field and O be the ring of integers. Then an indecomposable 2-lattice over O is isomorphic to one of following lattices:*

A_n $(n \geq 1)$, B_n $(n \geq 3)$, C_n $(n = 6, 7, 8)$, D_n $(n \geq 2)$, D'_4, F_n $(n = 2, 3, 4)$, *and* T_2.

Proof. Let L be an indecomposable 2-lattice. We shall prove the theorem by induction on the cardinal number of a minimal system of generators in L. If the number is one we have $L \cong A_1$. If the number is two then we have $L \cong A_2$, $L \cong D_2$, $L \cong F_2$ or $L \cong T_2$ by Lemma 6. Let $\{y, z, x_1, \cdots, x_n\}$ be a minimal system of generators in L, $n \geq 1$. We may assume that the lattice $Z = [z, x_1, \cdots, x_n]$ is indecomposable. Hence by the hypothesis of induction we have Z is isomorphic to one of the following:

A_{n+1}, B_{n+1} $(n \geq 2)$, C_{n+1} $(n = 5, 6, 7)$, D_{n+1}, F_{n+1} $(n = 1, 2, 3)$, T_2, and D'_4 $(n = 3)$.

We have $Z \not\cong C_8$, $Z \not\cong F_4$, $Z \not\cong T_2$ and $Z \not\cong D'_4$ by Remark 1 and Lemma 5. Thus we can assume that $B(x_i, x_j) = 1$ whenever $i \neq j$. So we may assume that the lattice $Y = [y, x_1, \cdots, x_n]$ is indecomposable by replacing y with $y - B(y, z)z$ if necessary. Apply Lemma 6 to the lattice Y. Then by exchanging y and z and rearranging $x_1, \cdots, x_n$ if necessary, we have

$$L = [y, z, x_1, \cdots, x_n] \cong \left(\begin{array}{cc|c} 2 & a & \\ a & 2 & F \\ \hline & & \begin{matrix} 2 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & 2 \end{matrix} \\ {}^tF & & \end{array}\right)$$

where $a \in \{0, \pm 1, \pm 2, \pm 3, \pm\zeta, \pm\bar{\zeta}\}$, F is a $2 \times n$ matrix whose row is one of the vectors specified in Lemma 6, and the part

$$\begin{pmatrix} 2 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & 2 \end{pmatrix}$$

denotes the matrix whose diagonal elements are 2 and the other elements are 1. Thus it suffices to treat the following cases (1)-(8), and so $a \neq \pm\sqrt{3}$ (since $d[y, z] \neq 1$ by Remark 1). Put

$$F_8 = \begin{pmatrix} \sqrt{2} & \sqrt{2} & \cdots & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & \cdots & \sqrt{2} \end{pmatrix}; \; F_9 = \begin{pmatrix} \sqrt{2} & \sqrt{2} & \cdots & \sqrt{2} \\ 0 & 1 & \cdots & 1 \end{pmatrix}, \; n \geq 3;$$

$$F_{10} = \begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} & \cdots & \sqrt{2} \\ 0 & 0 & 1 & \cdots & 1 \end{pmatrix}, \; 7 \geq n \geq 5; \; F_{11} = \begin{pmatrix} \zeta \\ \zeta \end{pmatrix}, \; n = 1;$$

$$F_{12} = \begin{pmatrix} \zeta & \zeta \\ \zeta & \zeta \end{pmatrix}, \; n = 2.$$

For $F_1, \cdots, F_7$, see Proof of Theorem A. We have $n = 5, 6$ in case that

$F \in \{F_3, F_4, F_5, F_6, F_7, F_{10}\}$ by Remark 1 and Lemma 5 since $dC_8 = 1$.

(1) If F has a row whose elements are 1, then the theorem holds by Lemma 6.

(2) If $F \in \{F_2, F_4, F_5\}$, then we have

$$Q(4y + (2 - 3a)z + (2 - a)x_1 + 2(a - 2)x_n) = 4(2 - a)(2 + 3a),$$

which is zero or totally positive. Hence $a \in \{0, 1, \zeta, \bar{\zeta}\}$. If $a = 0$ then $Q(y + z + x_1 - x_{n-1} - x_n) = 0$, which is a contradiction. If $a = \zeta$ or $a = \bar{\zeta}$ then L contains the lattice $[y, z, x_1, x_n]$ whose discriminant is ζ^2 or $\bar{\zeta}^2$. This is a contradiction by Remark 1. If $a = 1$ then we have $L \cong C_{n+2}$ similarly as in (4), (8) and (9) of the proof of Theorem A.

(3) If F $\{F_1, F_3, F_6, F_7\}$, then we have

$$Q(4y - (3a + 1)z - (a + 3)x_1 + 2(a + 1)x_2) = 4(1 - a)(5 + 3a),$$

which implies $a \in \{0, 1, -1, -\zeta, -\bar{\zeta}\}$. If $a = 1$ then we obtain $Q(y - z - x_1 + x_2) = 0$ which is absurd in view of Remark 3. If $a = 0$ then $L \cong C_{n+2}$ similarly as in (3), (5) and (6) of the proof of Theorem A. If $a = -1$ then we have $Q(y + z - x_n) = 0$, a contradiction. If $a = -\zeta$ or $a = -\bar{\zeta}$ then L contains the lattice $[y, z, x_1, x_2]$ whose discriminant is $\bar{\zeta}^2$ or ζ^2, which is impossible by Remark 1.

(4) If $F = F_8$, then we have

$$Q(2y - (2 + (n - 1)(a - 2))z + \sqrt{2}(a - 2)(x + \cdots + x_n))$$
$$= -2(a - 2)(4 + (n + 1)(a - 2)),$$

which implies that $a = 0$ with $n = 1$ or $a = 1$ with $n = 1, 2, 3$. If $a = 0$ and $n = 1$, then we have $Q(y + z - \sqrt{2}x_1) = 0$, a contradiction. If $a = 1$ and $n = 1$, then we have $L \cong D_3$. If $a = 1$ and $n = 2$, then we have $L \cong D_4'$. If $a = 1$ and $n = 3$, then we have $Q(\sqrt{2}y + \sqrt{2}z - x_1 - x_2$

$- x_3) = 0$, a contradiction.

(5) If $F = F_9$, then we have

$$Q(4y - (2\sqrt{2} + (n - 1)(a - \sqrt{2}))z - (2\sqrt{2} + (n - 3)(a - \sqrt{2}))x_1 + 2(a - \sqrt{2})(x_2 + \cdots + x_n))$$
$$= -4(a - \sqrt{2})(4\sqrt{2} + (n - 1)(a - \sqrt{2})),$$

which implies that $a = 0$ with $n \leq 3$ or $a = \sqrt{2}$. If $a = 0$ and $n = 3$ then we have $Q(\sqrt{2}y + z - x_2 - x_3) = 0$, a contradiction. If $a = \sqrt{2}$, then we have $Q(\sqrt{2}y - z - x_1) = 0$, a contradiction.

(6) If $F = F_{10}$, then by considering the lattice $[y, z, x_2, \cdots, x_n]$ we see that such a lattice does not exist since $n - 1 = 4, 5$ by (5).

(7) If $F = F_{11}$, then we have $Q(y + z - \zeta x_1) = 2 - 2\zeta + 2a$, which is totally positive. Hence we have $a = 1$ or $a = \zeta$, which implies $L \cong F_3$.

(8) If $F = F_{12}$, then we have $Q(y + z - x_1 - x_2) = 10 + 2a - 8\zeta$, which implies $a = \zeta$, and hence $L = [z, \zeta z - y, x_1, x_2] \cong F_4$.

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