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The Augmentation Subgroup of a Pretzel Link

By Taizo KANENOBU

1. Introduction

Let $L \subset S^3$ be a tame oriented link and let $G = \pi_1(S^3-L)$. If we define a homomorphism θ from G to the additive group of integers, Z, by sending the homotopy class of any loop to the sum of the linking numbers of this loop with the various components L_i of L, then the kernel of θ is called the *augmentation subgroup* of L. If L is a knot, then the augmentation subgroup is just the commutator subgroup ([1], [6]).

In the following we do not distinguish exactly between knots and links. Therefore by a link (with μ components) is meant a knot or link according as $\mu = 1$ or > 1. Moreover the reduced Alexander polynomial [2] will be denoted by $\Delta(t)$. If the link is a knot, this is just the Alexander polynomial.

In Section 3, we will study the structure of the augmentation subgroup of a pretzel link (see Section 2). From this result, in Section 4, we will define a p-surface, which is an extension of an s-surface in [5], and will prove that the link which spans a p-surface has a free augmentation subgroup of finite rank. Then we will give some examples of links which span p-surfaces.

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2. Pretzel links and surfaces

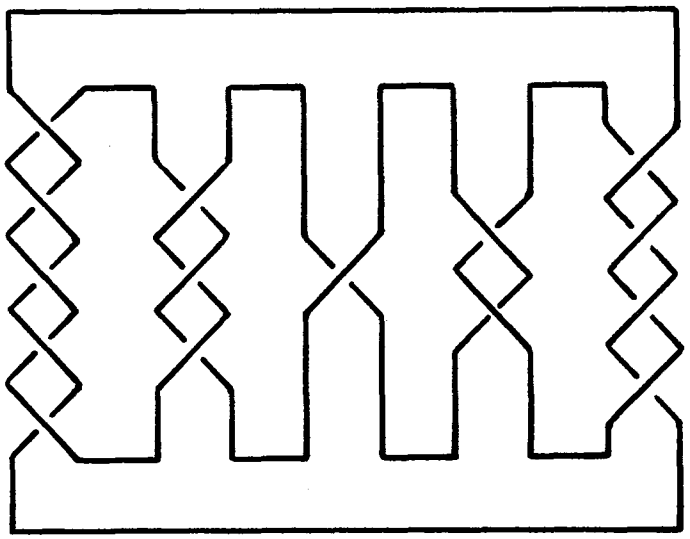
For $(p_1,\cdots,p_n)$ an n-tuple of integers, let us consider a surface $F(p_1,\cdots,p_n)$ in S^3, as is shown in Figure 1, consisting of two disks joined by n twisted but unknotted vertical bands, where the i^{th} band in order has p_i half-twists, right- or left-handed according to the sign of p_i. Let $K(p_1,\cdots,p_n)$ be the link formed by the boundary of $F(p_1,\cdots,p_n)$. We will call the link of this form a *pretzel link*, and call the surface a *pretzel surface*. ([10])

Let us say that $(q_1,\cdots,q_n)$ is a *cyclic rearrangement* of $(p_1,\cdots,p_n)$ if there is some k such that $q_i = p_{i+k}$ for all i interpreting the subscripts modulo n. Obviously two links $K(p_1,\cdots,p_n)$ and $K(q_1,\cdots,q_n)$ are equivalent.

Figure 2 illustrates an equivalence between the links $K(\cdot\cdot,p,1,\cdot\cdot)$ and $K(\cdot\cdot,1,p,\cdot\cdot)$. More generally, any p_i equal to 1 or -1 can be permuted freely in the n-tuple without changing the equivalence class of the associated link.

Definition 1. Two pretzel links $K(p_1,\cdots,p_n)$ and $K(q_1,\cdots,q_n)$ are *p-equivalent*, denoted by $K(p_1,\cdots,p_n) \sim K(q_1,\cdots,q_n)$, iff $(q_1,\cdots,q_n)$ can be changed to $(p_1,\cdots,p_n)$, $(-p_1,\cdots,-p_n)$, $(p_n,\cdots,p_1)$ or $(-p_n,\cdots,-p_1)$ by a cyclic rearrangement and some permutations of some q_i's which are equal to 1 or -1.

A pretzel link $K(p_1,\cdots,p_n)$ is said to be *degenerated* if there are p_i and p_j which are equal to 1 and -1 respectively. This pretzel link is clearly equivalent to $K(p_1,\cdot\cdot,\hat{p}_i,\cdot\cdot,\hat{p}_j,\cdot\cdot,p_n)$ ($\hat{p}_i$ means that p_i is omitted).



$K(5,-3,-1,2,-4)$

Figure 1

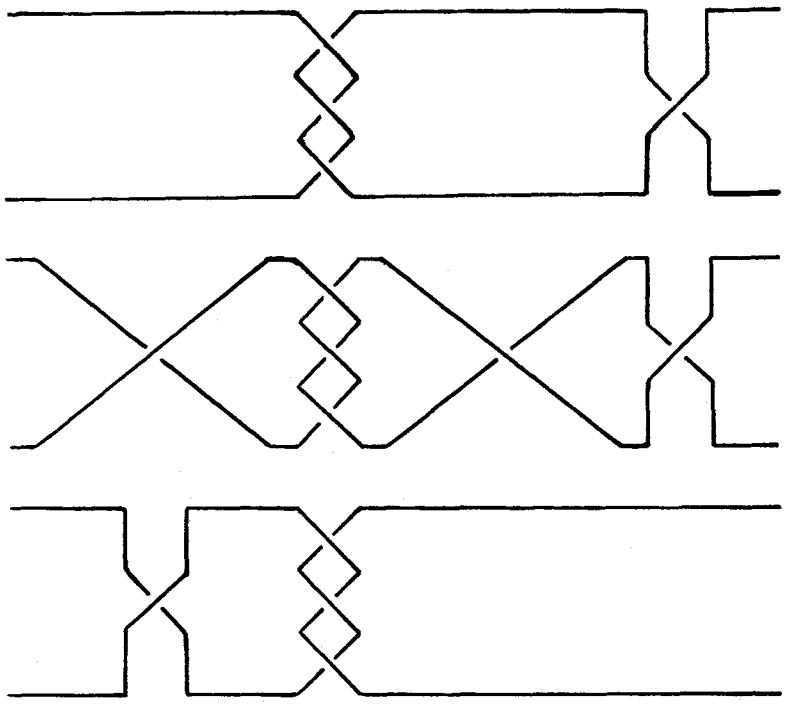


Figure 2

In this paper we will consider only an orientable pretzel surface, that is, all the p_i have the same parity, and consider the pretzel link which spans an orientable pretzel surface.

Definition 2. A pretzel surface $F = F(p_1,\cdots,p_n)$ $(n \geq 2)$ is a *primitive p-surface* if (i) for $n \geq 3$, the pretzel link $\partial F = K(p_1,\cdots,p_n)$ is not degenerated and its augmentation subgroup is free of finite rank, (ii) for $n = 2$, $F = F(1,1)$ or $F(-1,-1)$.

Remark. If the augmentation subgroup H of a link L is finitely generated, then L is *fibered* ([9]), and H is free of finite rank ([1], [6]).

3. Primitive p-surfaces

Let K be $K(1+2p_1, 1+2p_2,\cdots,1+2p_{n+1})$ $(n = 2,3,\cdots)$ which is not degenerated and let H be its augmentation subgroup. Let F be $F(1+2p_1, 1+2p_2,\cdots,1+2p_{n+1})$ which spans K and let U be an open regular neighborhood of the interior of F in S^3. The boundary of $S^3 - U$ is the union of two surfaces $F^{\#}$ and F^{b} whose intersection is K and which span K. The sets F and $S^3-(U \cup K)$ are strong deformation retracts of $cl(U)$ and S^3-F respectively. We define a homomorphism $\#$ by composition of mappings as shown in the diagram:

$$\begin{array}{ccccc} \pi_1(F^{\#}) & \longrightarrow & \pi_1(S^3 - U) & \xrightarrow{\simeq} & \pi_1(S^3 - (U \cup K)) \\ \simeq \downarrow d & & & & \downarrow \simeq \\ \pi_1(F) & & \xrightarrow{\qquad\#\qquad} & & \pi_1(S^3 - F) \end{array} .$$

(All mappings are induced by inclusion except for d and $\#$.)

$\pi_1(F)$ and $\pi_1(S^3 - F)$ are both free of rank n. We select bases: $\{x_1,\cdots,x_n\}$ for $\pi_1(F)$ and $\{a_1,\cdots,a_n\}$ for $\pi_1(S^3 - F)$, as shown in Figure 3.

The mapping $\#:\pi_1(F) \to \pi_1(S^3 - F)$ is given by

$$\begin{aligned}
x_1^{\#} &= (\bar{a}_2 a_1)^{1+p_2} a_1^{p_1}, \\
x_2^{\#} &= (\bar{a}_3 a_2)^{1+p_3} (\bar{a}_2 a_1)^{-p_2}, \\
&\vdots \\
x_{n-1}^{\#} &= (\bar{a}_n a_{n-1})^{1+p_n} (\bar{a}_{n-1} a_{n-2})^{-p_{n-1}}, \\
x_n^{\#} &= a_n^{1+p_{n+1}} (a_n a_{n-1})^{-p_n},
\end{aligned}$$

where $\bar{a}_i$ means a_i^{-1}. Then the Seifert matrix V of K is of the form:

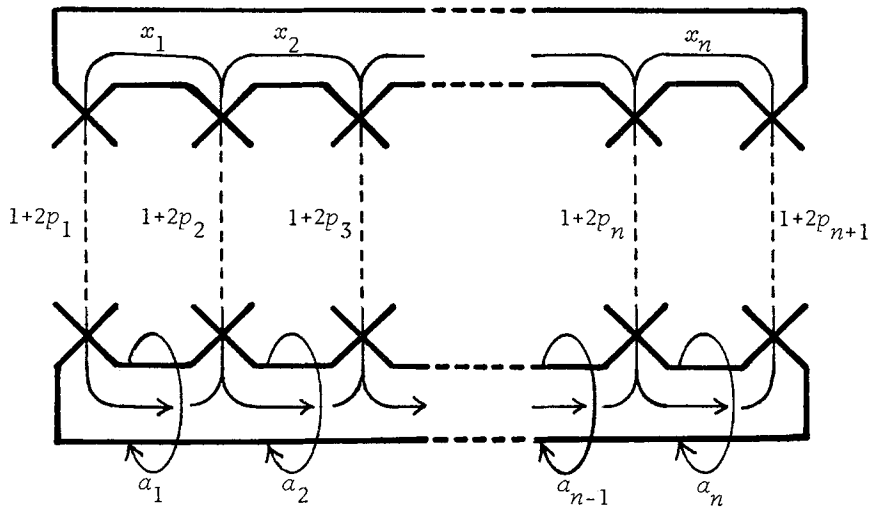


Figure 3

$$V = \begin{pmatrix} 1+p_1+p_2 & -1-p_2 & 0 & 0 & & \\ -p_2 & 1+p_2+p_3 & -1-p_3 & 0 & & \\ 0 & -p_3 & 1+p_3+p_4 & -1-p_4 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & -p_{n-1} & 1+p_{n-1}+p_n & -1-p_n & \\ & & 0 & -p_n & 1+p_n+p_{n+1} \end{pmatrix},$$

and the reduced Alexander polynomial $\Delta(t)$ is equal to $det(V - tV^{\mathsf{T}})$, where V^{T} is the transposed matrix of V.

Moreover by a straight induction on n, we have

$$\Delta(0) = det\, V = \prod_{i=1}^{n+1} (p_i+1) - \prod_{i=1}^{n+1} p_i \ .$$

(This formula has already been seen in [10] (3.1).)

It is easy to see that $\{x_1^{\#},\cdots,x_n^{\#}\}$ is a basis for a free subgroup of $\pi_1(S^3 - F)$, because much cancellation cannot occur among them. Therefore $\#$ is injective.

The next proposition is a generalization of the result in [3].

Proposition 3. *Let K be $K(1+2p_1, 1+2p_2,\cdots,1+2p_{n+1})$ $(n = 2,3,\cdots)$, which is not degenerated and let H be its augmentation subgroup. If $K \sim K(\underbrace{1,\cdots,1}_{i},-3,\cdots,-3)$ for $i \neq 0$, then H is free of rank n. Otherwise H is a nontrivial free product with amalgamation of the form:*

$$\cdots \underset{F_n}{*} F_n \underset{F_n}{*} F_n \underset{F_n}{*} \cdots ,$$

where F_n is a free group of rank n.

Proof. Suppose that H is free of rank n. Then $\#$ is an isomorphism onto ([1],[6]). Therefore $< a_1,\cdots,a_n \ ; \ x_2^{\#} =\cdots= x_n^{\#} = 1 >$ presents an

infinite cyclic group. Now if we adjoin the relations $b = a_2 = \cdots = a_n$, then the group becomes $< a , b ; (ba_1)^{p_2} = b^{1+p_{n+1}} = 1 >$, which is a free product, $Z_{|p_2|} * Z_{|1+p_{n+1}|}$. Since this group is the factor group of the infinite cyclic group, we have $|p_2| = 1$ or $|1+p_{n+1}| = 1$.

Moreover, if H is free of finite rank, then a slight modification of results in [7] also shows that the rank of H must be the degree of $\Delta(t)$, that is n, and that $|\Delta(0)| = 1$. Thus we obtain $|\det V| = |\prod_{i=1}^{n+1} (1+p_i) - \prod_{i=1}^{n+1} p_i| = 1$.

If $p_{n+1} = 0$, $|\det V| = |\prod_{i=1}^{n+1} (1+p_i)| = 1$. It follows that $p_i = 0$ or -2 for all i. If $p_2 = -1$, $|\det V| = |\prod_{i=1}^{n+1} p_i| = 1$. It follows that $p_i = 1$ or -1 for all i.

On the other hand, by cyclic rearrangements of $(1+2p_1,\cdots,1+2p_{n+1})$, we have

$$
\begin{aligned}
&(1) \quad |p_3| = 1 \quad \text{or} \quad |1+p_1| = 1,\\
&(2) \quad |p_4| = 1 \quad \text{or} \quad |1+p_2| = 1,\\
&\quad \vdots\\
&(n) \quad |p_1| = 1 \quad \text{or} \quad |1+p_n| = 1.
\end{aligned}
$$

If one of the numbers $p_1,\cdots,p_{n+1}$ is 0 (resp. -1), then the others must equal 0 or -2 (resp. 1 or -1). Thus let us suppose that any of them do not equal 0 or -1. Then we have

$$
\begin{aligned}
&(0') \quad p_2 = 1 \quad \text{or} \quad p_{n+1} = -2,\\
&(1') \quad p_3 = 1 \quad \text{or} \quad p_1 = -2,\\
&\quad \vdots\\
&(n') \quad p_1 = 1 \quad \text{or} \quad p_n = -2.
\end{aligned}
$$

Case i) n is even. If $p_2 = 1$, then $p_4 = 1$ by (2'), then $p_6 = 1$

by (4'),$\cdots$, then $p_{n-2} = 1$ by (n-4'), then $p_n = 1$ by (n-2'). So we have $p_1 = 1$ by (n'), then $p_3 = 1$ by (1'), $\cdots$, then $p_{n-1} = 1$ by (n-3'), then $p_{n+1} = 1$ by (n-1'). Thus we have $p_1 = p_2 = \cdots = p_{n+1} = 1$. Since $|\Delta(0)| = |\det V| = |2^{n+1}-1^{n+1}| \neq 1$, H is not free of finite rank.

If $p_2 \neq 1$, then $p_{n+1} = -2$. In the same way, we have $p_i = -2$ for all i. Thus H is not free of finite rank.

Case ii) n is odd. The same method gives us that

(a) $p_2 = p_4 = \cdots = p_{n+1} = 1$ or $p_2 = p_4 = \cdots = p_{n+1} = -2$,

and (b) $p_1 = p_3 = \cdots = p_n = 1$ or $p_1 = p_3 = \cdots = p_n = -2$.

Then the calculation of $\Delta(0)$ shows us that in any cases H is not free of finite rank.

Conversely, if $K \sim K(1,\cdots,1,-3,\cdots,-3)$, then it is easy to see that $\#$ is an isomorphism onto. Therefore H is free of rank n. This completes the proof of Proposition 3.

From now on we will consider another type of pretzel links. Let L be $K(2p_1,\cdots,2p_n)$ ($n = 3,4,\cdots$), where none of the p_i is equal to zero. Let F be $F(2p_1,\cdots,2p_n)$ which spans L. Let us define a homomorphism $\# : \pi_1(F) \to \pi_1(S^3 - F)$ as defined before. $\pi_1(F)$ and $\pi_1(S^3 - F)$ are both free of rank n-1. We select bases; $\{x_1, \cdots, x_{n-1}\}$ for $\pi_1(F)$ and $\{a_1, \cdots, a_{n-1}\}$ for $\pi_1(S^3 - F)$ as shown in Figure 4. The mapping $\#$ is given by

$$x_1^{\#} = a_1^{p_1}(\bar{a}_1 a_2)^{-p_2},$$

$$x_2^{\#} = (\bar{a}_1 a_2)^{p_2}(\bar{a}_2 a_3)^{-p_3},$$

$$\vdots$$

$$x_{n-2}^{\#} = (\bar{a}_{n-3}a_{n-2})^{p_{n-2}}(\bar{a}_{n-2}a_{n-1})^{-p_{n-1}},$$
$$x_{n-1}^{\#} = (\bar{a}_{n-2}a_{n-1})^{p_{n-1}}a_{n-1}^{p_n}.$$

Then the Seifert matrix V of L is the following:

$$V = \begin{pmatrix} p_1+p_2 & -p_2 & 0 & 0 & & & \\ -p_2 & p_2+p_3 & -p_3 & 0 & & & \\ 0 & -p_3 & p_3+p_4 & -p_4 & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & -p_{n-1} & 1+p_{n-1}+p_n & -1-p_n & \\ & & & 0 & -p_n & 1+p_n+p_{n+1} & \end{pmatrix}.$$

Note that this matrix is symmetric; $V^T = V$. Then the reduced Alexander polynomial can be computed directly as follows:

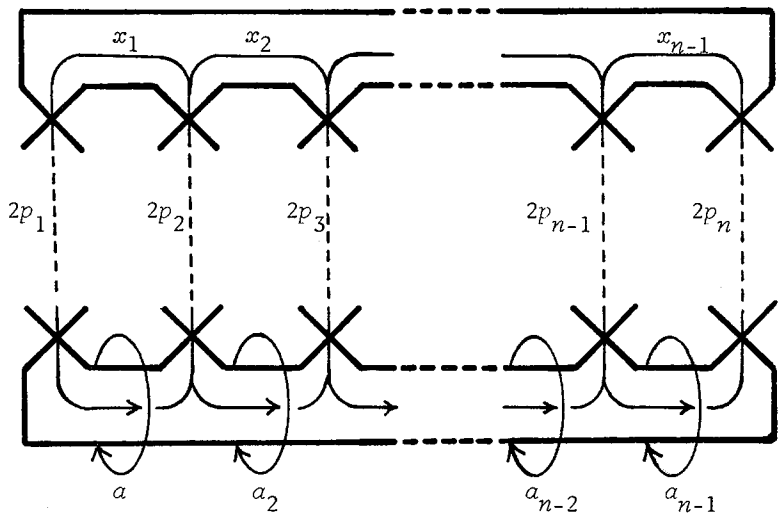


Figure 4

$$\Delta(t) = det(V - tV^{T}) = det\ (V - tV) = det\ V(E - tE)$$
$$= detV\cdot(1-t)^{n-1},$$

where E is the $(n\text{-}1, n\text{-}1)$-unit matrix. We can obtain $det\ V$ by an induction, that is, $det\ V = p_1\cdots p_n \sum_{i=1}^{n} \frac{1}{p_i}$. Thus we have

$$\Delta(t) = p_1\cdots p_n \sum_{i=1}^{n} \frac{1}{p_i}(1-t)^{n-1}.$$

If $\Delta(t) = 0$, H is infinitely generated. ([7]) If $\Delta(t) \neq 0$, L is *strongly indecomposable* [1], that is, L has an orientable spanning surface of maximal Euler characteristic which is connected. Let F' be such a spanning surface of L. Then the Euler characteristic of F' must be less than or equal to $2\text{-}n$. On the other hand, the Euler characteristic of F is just $2\text{-}n$. Thus F is a spanning surface of maximal Euler characteristic. Hence $\#$ is injective by (4.1) in [1].

Lemma 4. *If $\#$ is an isomorphism onto, then at most one of the $|p_i|$ is not equal to* 1.

Proof. Since $\#$ is isomorphic onto, $< a_1,\cdots,a_{n-1} ; x_1^{\#} =\cdots= x_{n-2}^{\#} = 1 >$ presents an infinite cyclic group. If we let $b_1 = a_1$, $b_2 = \bar{a}_1 a_2$, $\cdots$, $b_{n-1} = \bar{a}_{n-2}a_{n-1}$, this group becomes $< b_1,\cdots,b_{n-1} ; b_1^{p_1} = b_2^{p_2} =\cdots= b_{n-1}^{p_{n-1}} >$. Therefore the factor group of the infinite cyclic group is a free procuct $Z_{|p_1|} * Z_{|p_2|} * \cdots * Z_{|p_{n-1}|}$. Thus we obtain that at most one of the $|p_1|$, $|p_2|,\cdots, |p_{n-1}|$ is not equal to 1. If $|p_1| = |p_2| =\cdots= |p_{n-1}| = 1$, the proof has already finished. If $|p_i| \neq 1$ for some i $(1 \leq i \leq n\text{-}1)$, we can obtain that $|p_n| = 1$ by some cyclic rearrangements. This completes the proof.

Lemma 5. *Suppose that # is an isomorphism onto. If n is odd, then the numbers of 1 and -1 in the p_i are both $\frac{n-1}{2}$. If n is even, then the number of 1 (resp. -1) is $\frac{n}{2}$, that of -1 (resp. 1) is $\frac{n}{2}$ -1 and that of -2 (resp. 2) is 1.*

Proof. Let $|p_1| = \cdots = |p_{n-1}| = 1$ and $p_n = p$. Let α and β be the numbers of 1 and -1 in the $p_1, \cdots, p_{n-1}$ respectively. Then $\alpha+\beta = n-1$. Because # is isomorphic onto, $|\Delta(0)| = 1$. On the other hand $\Delta(0) = p_1 \cdots p_{n-1} + p_1 \cdots p_{n-1} p \sum_{i=1}^{n-1} \frac{1}{p_i} = (-1)^{\beta} + (-1)^{\beta} p(\alpha-\beta)$. Thus $p(\alpha-\beta) = 0$ or -2.

If n is odd, $\alpha-\beta = 0$ or 2 or -2. When $\alpha-\beta = 0$, then $\alpha = \beta = \frac{n-1}{2}$; when $\alpha-\beta = 2$, then $\alpha = \frac{n+1}{2}$, $\beta = \frac{n-3}{2}$ and $p = -1$; when $\alpha-\beta = -2$, then $\alpha = \frac{n-3}{2}$, $\beta = \frac{n+1}{2}$ and $p = 1$.

If n is even, $\alpha-\beta = 1$ or -1. When $\alpha-\beta = 1$, then $\alpha = \frac{n}{2}$, $\beta = \frac{n}{2}-1$ and $p = -2$; when $\alpha-\beta = -1$, then $\alpha = \frac{n}{2}-1$, $\beta = \frac{n}{2}$ and $p = 2$.

The proof of Lemma 5 is complete.

Proposition 6. *Let L be $K(2p_1, \cdots, 2p_n)$ $(n = 3,4,\cdots)$, where $p_i \neq 0$ for all i and let H be its augmentation subgroup. If $L \sim K(2,-2,\cdots, 2,-2,2p)$ when n is odd, or $L \sim K(2,-2,2,-2,\cdots,2,-4)$ when n is even, then H is free of rank $n-1$. Otherwise H is not finitely generated.*

Proof. Let $b_1 = a_1$, $b_2 = \bar{a}_1 a_2$, $b_3 = \bar{a}_2 a_3$, $\cdots\cdots$, $b_{n-1} = \bar{a}_{n-2} a_{n-1}$, then $\pi_1(S^3 - F)$ becomes $< b_1, \cdots, b_{n-1} ; >$. Besides let $y_1 = x_1$, $y_2 = x_1 x_2, \cdots$, $y_{n-1} = x_1 x_2 \cdots x_{n-1}$, then $\pi_1(F)$ becomes $< y_1, \cdots, y_{n-1} ; >$. The mapping $\# : \pi_1(F) \to \pi_1(S^3 - F)$ is given by

$$y_1^{\#} = b_1^{p_1} \bar{b}_2^{p_2},$$
$$y_2^{\#} = b_1^{p_1} \bar{b}_3^{p_3},$$

$$\vdots$$

$$y_{n-2}^{\#} = b_1^{p_1}\bar{b}_n^{p_{n-1}},$$

$$y_{n-1}^{\#} = b_1^{p_1}(b_1b_2\cdots b_{n-1})^{p_n}.$$

Suppose that H is free of rank n. Then $\#$ is isomorphic onto. By Lemma 5, we may assume $p_1 = 1$, $|p_2| = |p_3| = \cdots = |p_{n-1}| = 1$. We put $p_n = p$, $b_1 = b$ and we omit $\#$ from $y_i^{\#}$. Then $b_2 = (\bar{y}_1b)^{p_2}$, $b_3 = (\bar{y}_2b)^{p_3}$, $\cdots$, $b_{n-1} = (\bar{y}_{n-2}b)^{p_{n-1}}$. Thus $y_{n-1} = b\{b(\bar{y}_1b)^{p_2}(\bar{y}_2b)^{p_3}\cdots(\bar{y}_{n-2}b)^{p_{n-1}}\}^p$. Let $R = \bar{y}_{n-1}b\{b(\bar{y}_1b)^{p_2}(\bar{y}_2b)^{p_3}\cdots(\bar{y}_{n-1}b)^{p_{n-1}}\}^p$, then $\pi_1(S^3 - F) = \langle b, y_1, \cdots, y_{n-1} ; R = 1 \rangle$. On the other hand $\#$ is isomorphic onto, so b must be written by a word in $y_1, y_2, \cdots, y_{n-1}$, say, W. If let $S = \bar{b}W$, $\pi_1(S^3 - F)$ can be presented by $\langle b, y_1, \cdots, y_{n-1} ; S = 1 \rangle$. By Theorem 4.11 in [4], R and S^{ε} are conjugate in the free group on $b, y_1, \cdots, y_{n-1}$ for $\varepsilon = 1$ or -1. Thus b or $\bar{b}$ must appear in R only one time. Hence we can obtain the following results.

If n is even, then $p_2 = p_4 = \cdots = p_{n-2} = -1$, $p_3 = p_5 = \cdots = p_{n-1} = 1$ and $p = -2$. If n is odd and $|p| \geq 2$, then $p_2 = p_4 = \cdots = p_{n-1} = -1$ and $p_3 = p_5 = \cdots = p_{n-2} = 1$. If n is odd and $|p| = 1$, then $L \sim K(2,-2,\cdots,2,-2,2)$.

Conversely if $L \sim K(2,-2,\cdots,2,-2,2p)$ or $K(2,-2,\cdots,2,-4)$, then it is easy to see that $\#$ is isomorphic onto.

This completes the proof of Proposition 6.

Propositions 3 and 6 imply immediately the following:

Theorem 7. *A pretzel surface $F = F(p_1,\cdots,p_n)$ $(n \geq 2)$ is a primitive p-surface, iff the pretzel link $\partial F = K(p_1,\cdots,p_n)$ is p-equivalent to one*

of the following pretzel links;

for $n \geq 3$, $K(\underbrace{1,\cdots,1}_{i},-3,\cdots,-3)$ $(1 \leq i \leq n)$,

$K(2,-2,\cdots,2,-2,2p)$ $(p \neq 0)$, when n is odd,

$K(2,-2,\cdots,2,-2,2,-4)$, when n is even,

for $n = 2$, $K(1,1)$.

Remark. Especially $F(1,1,\cdots,1)$ (or $F(-1,-1,\cdots,-1)$) is called a *primitive s-surface* in [5].

4. p-surfaces

Consider two primitive p-surfaces $F = F(p_1,\cdots,p_n)$ and $F' = F(q_1,\cdots,q_m)$ in S^3. Take two disks, say, D and D', from each F and F' and identify them so that the resulting orientable surface $P = F \cup F'$ spans a link, and that $P\text{-}F$ and $P\text{-}F'$ are *separated*, i.e. there exists a 2-sphere S in S^3 such that $S \cap P = D = D'$ and each component of $S^3\text{-}S$ contains points of $P\text{-}D$. P consists of three disks and $n+m$ disjoint bands. P will be called a *p-surface*. Similarly, given two p-surfaces, we can construct an orientable surface spanning a link by identifying two disks, one from each of the p-surfaces. In general, by a p-surface is meant an orientable surface obtained from a number of primitive p-surfaces by identifying their disks in this manner.

A p-surface is an extension of an s-surface in [5]. If a p-surface P is constructed from only primitive s-surfaces, then P is called an *s-surface* in [5]. Therefore a slight modification of Theorem 4.3 in [5] implies the following:

Theorem 8. *Let P be a p-surface and let d be the degree of the reduced Alexander polynomial $\Delta(t)$ of a link ∂P. Then the augmentation subgroup of ∂P is free of rank d and $\Delta(0) = \pm 1$.*

In [5], it is remarked that 9_{44} in [8] cannot be spanned by any s-surface, but 9_{44} can be spanned by a p-surface. Thus 9_{44} has a free commutator subgroup. Moreover we can show the following:

Theorem 9. *Any prime knot of 10 crossings or less in* [8] *for which $\Delta(0) = \pm 1$ can be spanned by a p-surface.*

Proof. For an alternating knot, we can span an s-surface by Lemma 3.1 in [5]. For a knot which has a non-alternating projection in [8], see Appendix I.

Remark. Let K be a prime knot of 10 crossings or less in [8] and H be its commutator subgroup. Then the following are equivalent.

(1) $\Delta(0) = \pm 1$.

(2) H is free.

(3) K is fibered.

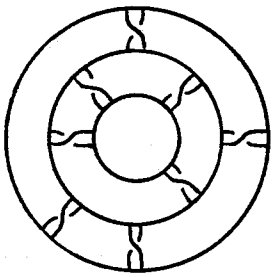
Also for links we will give some examples of spanning p-surfaces in Appendix II.

At the end of this paper, we raise a conjecture.

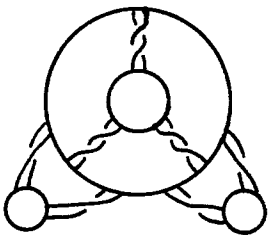
Conjecture 10. Any link, whose augmentation subgroup is free of finite rank, can be spanned by a p-surface.

Appendix I

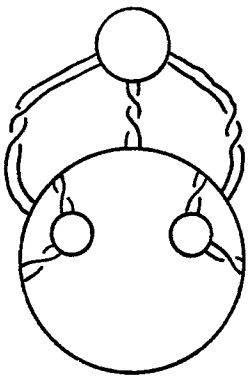
8_{19}



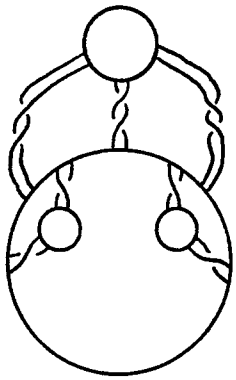
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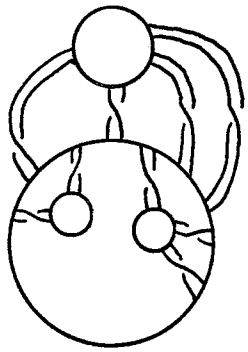
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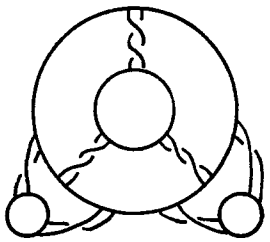
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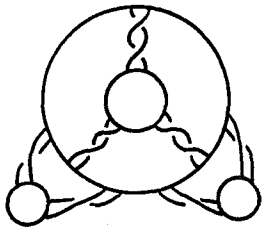
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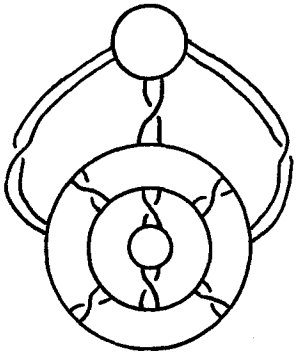
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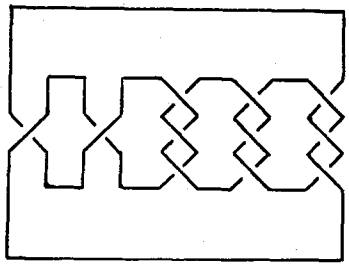
9_{45}



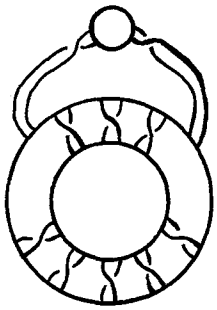
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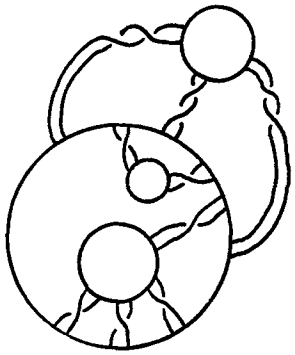
$9_{48} = K(-1,-1,3,3,3)$



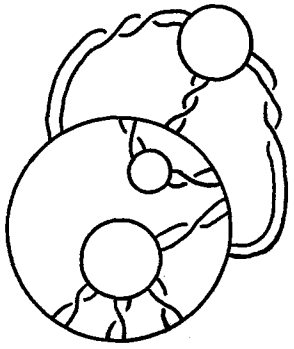
10_{124}



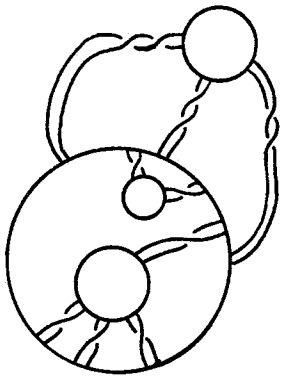
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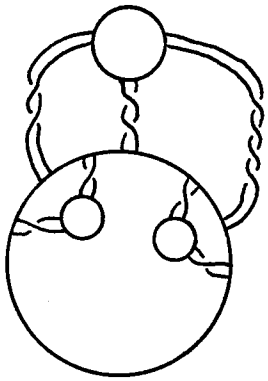
10_{126}



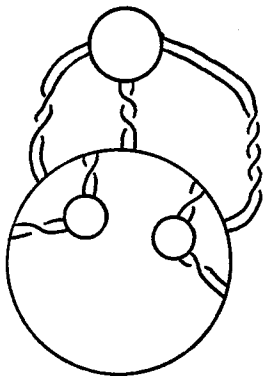
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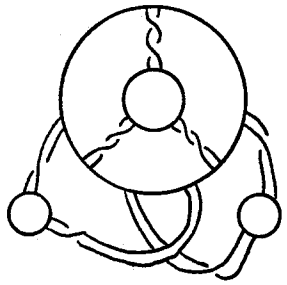
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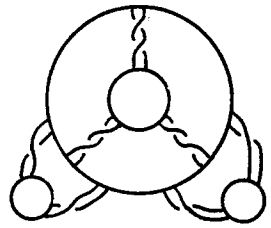
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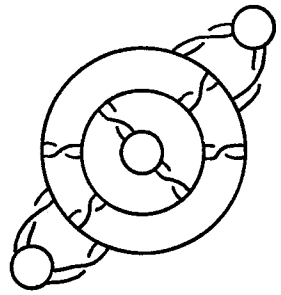
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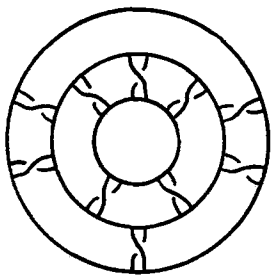
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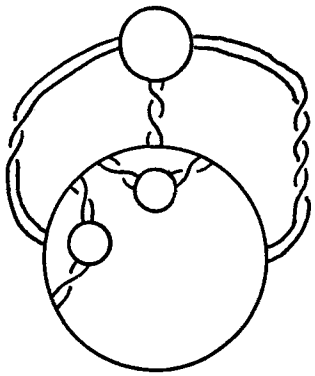
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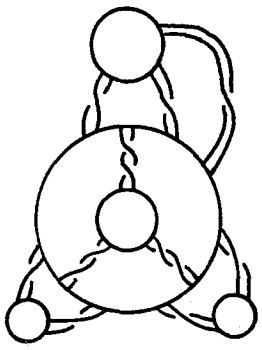
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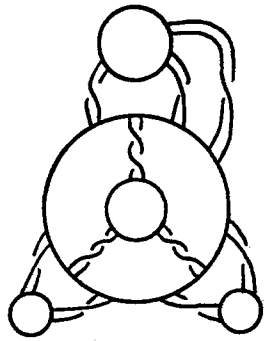
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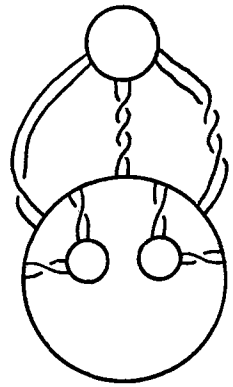
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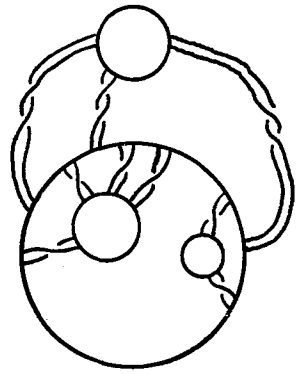
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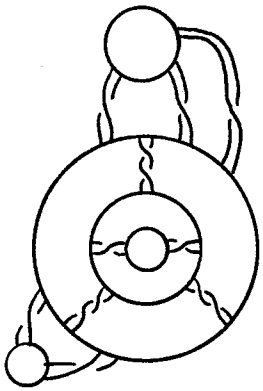
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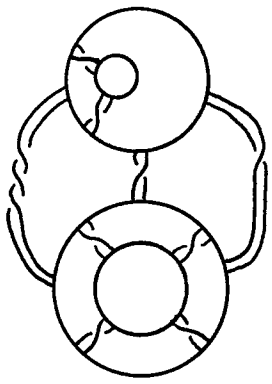
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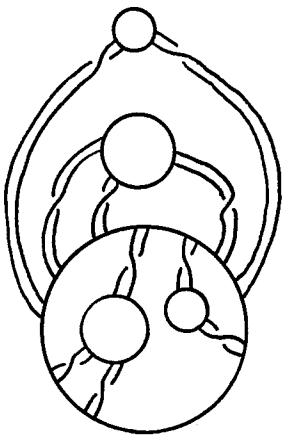
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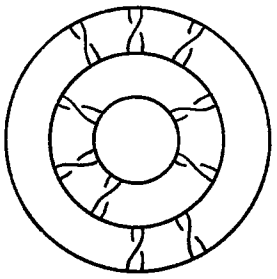
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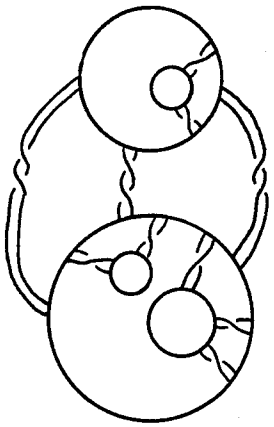
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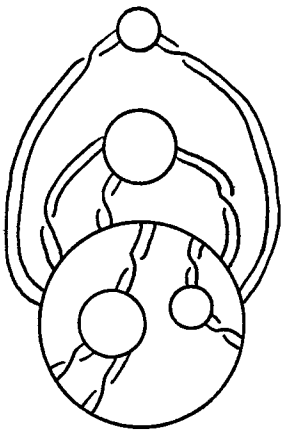
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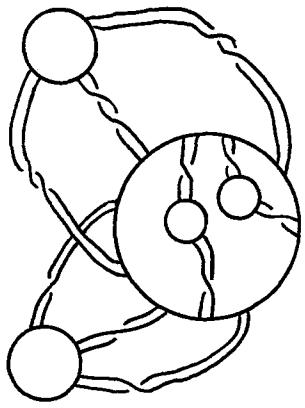
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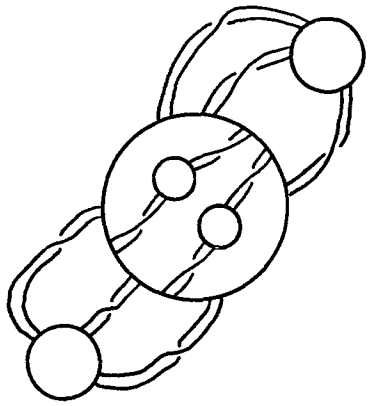
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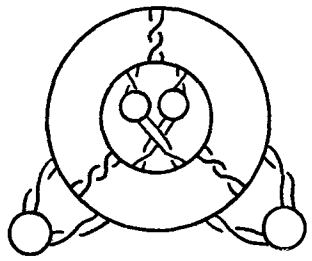
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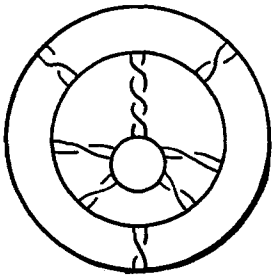
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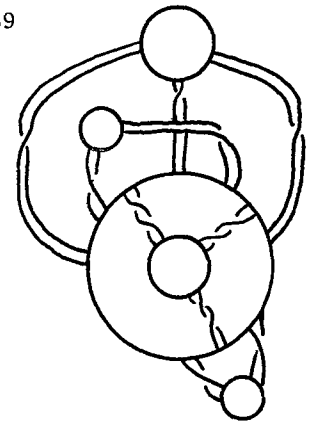
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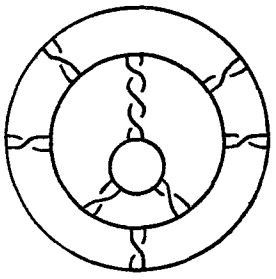
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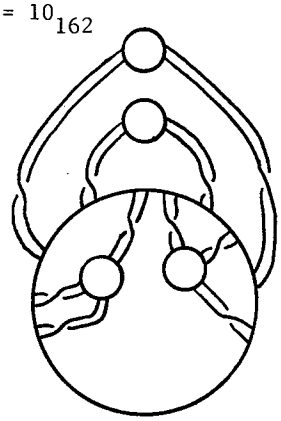
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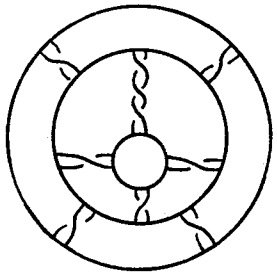
10_{160}



$10_{161} = 10_{162}$

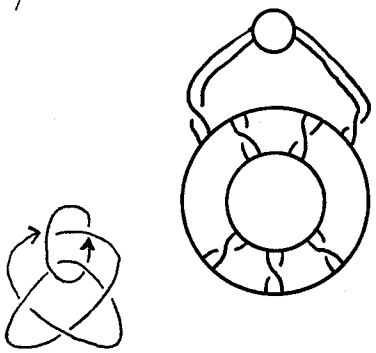


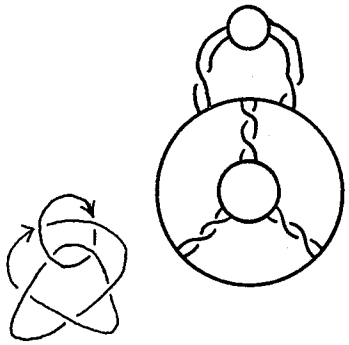
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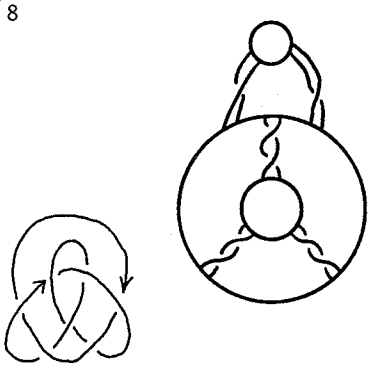
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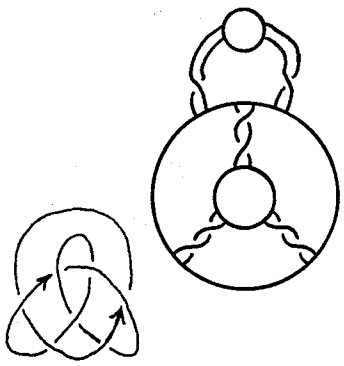
7_7^2





7_8^2





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