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Hosokawa, Fujitsugu

Maeda, Toru

Suzuki, Shin'ichi

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Numerical Invariants of Surfaces in 4-Space

Dedicated to Professor Kiyoshi Iséki on his 60th birthday

By Fujitsugu HOSOKAWA, Toru MAEDA and Shin'ichi SUZUKI

This is a continuation of Hosokawa-Kawauchi [4], where they have proposed a concept of unknotted surfaces in the Euclidean 4-space R^4 . In this paper we will introduce two numerical invariants of a surface in 4-space related to it and discuss primary topics of them. Throughout the paper, spaces and maps will be considered in the piecewise-linear category, and we will use the same definitions and notation as [4] unless otherwise stated.

1. Compositions of surfaces

By a *surface* we mean a closed, connected and orientable 2-manifold. For a surface F , $g(F)$ stands for the genus of F . We consider an embedded surface in the oriented 4-space R^4 or 4-sphere S^4 . We will always assume that F is *oriented* and *locally flat* in R^4 or S^4 .

Two surfaces F and F' in R^4 (or S^4) are said to be *of the same knot type*, iff there exists an orientation preserving homeomorphism ψ of R^4 (or S^4) onto itself such that $\psi(F) = F'$ and $\psi|_F$ is also orientation preserving. We denote the *knot type* of F in R^4 (or S^4) by $[F]$, so F

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is a representative of $[F]$. Those surfaces of the same knot type as the standard 2-sphere S^2 in R^4 (or S^4) are called *trivial*, and constitute the *trivial type* $[S^2]$.

Before stating our definitions of numerical invariants in Sections 2 and 3, we introduce here a concept of "composition" of surfaces, due to Mazur [8], Schubert [9], etc.

1.1. Definition. Let F_1 and F_2 be surfaces in S^4 , and let B_1^4 and B_2^4 be 4-cells in S^4 such that pairs $(B_1^4, B_1^4 \cap F_1)$ and $(B_2^4, B_2^4 \cap F_2)$ are homeomorphic to the standard cell-pair (D^4, D^2) . Then, the *composition* $F_1 \# F_2$ of surfaces F_1 and F_2 is a new surface F in S^4 obtained by matching the boundaries $\partial(S^4 - \text{Int}B_1^4)$ and $\partial(S^4 - \text{Int}B_2^4)$ using an orientation reversing homeomorphism ζ such that $\zeta\partial(F_1 - \text{Int}B_1^4) = \partial(F_2 - \text{Int}B_2^4)$ and $\zeta|_{F_1 - \text{Int}B_1^4}$ is also orientation reversing.

conversely, we will say that $F_1 \# F_2$ is a *decomposition* for F , and we denote $F = F_1 \# F_2$.

By the homogeneity theorem of Newman-Gugenheim (cf. Gugenheim [3]), the operation $\#$ is well-defined up to knot type, and also it is associative and commutative. So, the *composition*

$$[F] = [F_1] \# [F_2] = [F_1 \# F_2]$$

of two knot types of surfaces is well-defined.

For surfaces in the 4-space R^4 , we can also define, in a natural way, a *composition* $\#$ (and a *decomposition*).

1.2. Theorem. (Mazur [8], cf. Suzuki [11, §10]) *The set F of all knot types of surfaces in S^4 (or R^4) forms a commutative semi-group under the composition $\#$. In particular, the trivial type $[S^2]$ is a unit.*

The set K_2 of all knot types of a 2-sphere in S^4 (or R^4) is also a commutative semi-group under the composition $\#$. The natural inclusion of K_2 in F is injective.

1.3. Definition. A knot type $[F]$ of a surface F in S^4 (or R^4) is said to be *prime*, iff $[F] \neq [S^2]$ and there exist no decompositions $[F] = [F_1] \# [F_2]$ with both $[F_1]$ and $[F_2]$ non-trivial.

We have the following natural problem.

1.4. Problem. (i) Does there exist a prime decomposition for any non-trivial knot type $[F]$?

(ii) If (i) is affirmative, is the prime decomposition unique ?

2. Knotting genus of a surface

In order to introduce our invariants, we use the following basic fact.

2.1. Proposition. (Gluck [2], cf. [5],[11]) For any surface F in R^4 (or S^4), there exists a compact, connected and oriented 3-manifold V^3 in R^4 (or S^4) with $\partial V^3 = F$.

We call such a 3-manifold V^3 a *Seifert manifold* for F .

2.2. Definition. (Hosokawa-Kawauchi [4]) A surface F in R^4 (or S^4) is said to be *unknotted*, iff there exists a Seifert manifold V^3 for F such that V^3 is a solid torus of genus $g(F)$. By a *solid torus of genus n* we mean a 3-manifold homeomorphic to a regular neighborhood in R^3 of a compact, connected 1-complex with Euler characteristic $1-n$.

2.3. Proposition. If a surface F in R^4 (or S^4) is unknotted, then the fundamental group $\pi_1(R^4 - F)$ (or $\pi_1(S^4 - F)$) is infinite cyclic.

If D is a 2-cell, then $N(D)$ will denote a 3-manifold homeomorphic to $D \times [-1, 1]$, where D corresponds to $D \times \{0\}$.

It is well-known (cf. Seifert-Threlfall [10]) that any compact, connected and orientable 3-manifold V^3 with non-empty connected boundary can be represented as $T \cup N(D_1) \cup \dots \cup N(D_k)$ where T is a solid torus, D_i is a 2-cell for each i , $N(D_i) \cap N(D_j) = \emptyset$ if $i \neq j$, and $N(D_i) \cap T = \partial N(D_i) \cap \partial T$ corresponds to $\partial D_i \times [-1, 1]$ in $N(D_i)$. This will be called a *Heegaard splitting* for V^3 , and $N(D_i)$ is called a *handle of index 2*. The genus of the splitting is the genus of T and the smallest possible genus of a Heegaard splitting for V^3 will be denoted by $Hg(V^3)$.

2.4. Definition. A surface F in R^4 (or S^4) is of *knotting genus* k , denoted by $Kg(F) = k$, iff there exists a Seifert manifold V^3 for F with $V^3 = T \cup N(D_1) \cup \dots \cup N(D_k)$, and for any Seifert manifold V'^3 for F with $V'^3 = T' \cup N(D'_1) \cup \dots \cup N(D'_h)$, it holds that $k \leq h$. The knotting genus is a knot type invariant, and by $Kg[F]$ we denote the *knotting genus* of $[F]$.

2.5. Proposition. For a surface F in R^4 (or S^4) of knotting genus k , let $V^3 = T \cup N(D_1) \cup \dots \cup N(D_k)$ be a Seifert manifold. Then it holds that

$$g(F) = Hg(V^3) - Kg(F).$$

2.6. Proposition. A surface F in R^4 (or S^4) is unknotted if and only if $Kg(F) = 0$.

2.7. Question. Suppose that $[F] = [F_1] \# [F_2]$. Does it hold that

$$Kg[F] = Kg[F_1] + Kg[F_2] \quad ?$$

An affirmative answer to Question 2.7 implies, of course, an affirmative answer to Problem 1.4 (i).

3. Unknotting number of a surface

By Proposition 2.1 and Definition 2.2, we have the following :

3.1. Theorem. (Hosokawa-Kawauchi [4, Theorem 2.3]) *For any surface F in R^4 (or S^4), there exists a finite number of 1-handles $B_1, \dots, B_u$ such that the surface $h^1(F; B_1, \dots, B_u)$, obtained from F by hyperboloidal transformations along 1-handles $B_1, \dots, B_u$, is an unknotted surface of genus $g(F) + u$.*

Proof. Let V^3 be a Seifert manifold for F with a Heegaard splitting $V^3 = T \cup N(D_1) \cup \dots \cup N(D_u)$. We may take $N(D_i) \cong D_i \times [-1, 1]$ as a 1-handle $B_i \cong [-1, 1] \times D^2$, $i=1, \dots, u$.

Now we define the unknotting number of F , which corresponds, in certain sense, to the unknotting number of the classical knots.

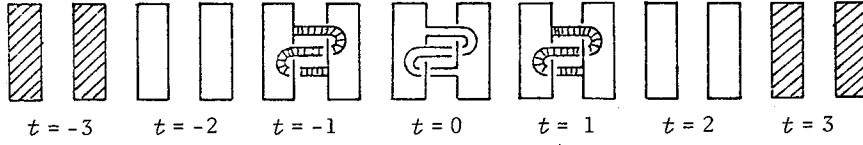
3.2. Definition. The *unknotting number*, $u(F)$, of a surface F in R^4 (or S^4) is the least number of 1-handles $B_1, \dots, B_u$ for F so that $h^1(F; B_1, \dots, B_u)$ is unknotted. The unknotting number is also a knot type invariant, and by $u[F]$ we denote the *unknotting number* of $[F]$.

3.3. Proposition. *For any surface F in R^4 (or S^4), it holds that*

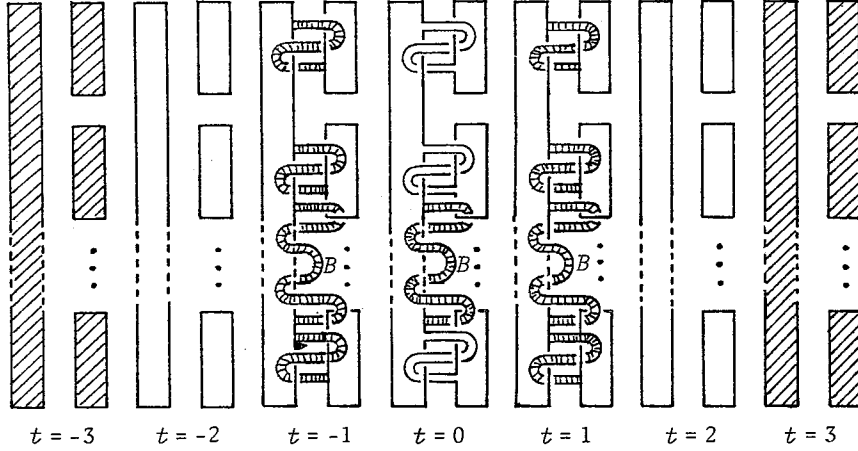
$$0 \leq u(F) \leq Kg(F) < \infty .$$

3.4. Theorem. *For arbitrary non-negative integers n and u , there exists a surface F in R^4 (or S^4) with $g(F) = n$ and $u(F) = u$.*

Proof. It suffices to show that there exists a 2-sphere, say S_u , in R^4 with $u(S_u) = u$. The following 2-sphere S_1 illustrated in Fig.1 shows the case $u = 1$, which is given by Hosokawa-Kawauchi [4, Fig.3 and 2.9].

Fig. 1 : S_1 in R^4

Using this 2-sphere S_1 , we construct a required 2-sphere S_u for any $u \geq 2$ as a composition of u copies of S_1 ; $S_u = S_1 \# \cdots \# S_1$. From

Fig. 2 : S_u in R^4 , and a 1-handle B

the definition of the unknotting number, it is easy to see that :

$$u(S_u) \leq u(S_1) + \cdots + u(S_1) = u.$$

So, we will show that $u(S_u) \geq u$.

The fundamental group $G_u = \pi_1(R^4 - S_u)$ has the following presentation :

$$\begin{aligned} G_u &= \langle x, y_1, \dots, y_u \mid xy_i x = y_i xy_i \ (i=1, \dots, u) \rangle \\ &= \left\langle x, y_1, \dots, y_u, \right. \left. \begin{array}{l} y_i = a_{i0} x, \ a_{i1} = x a_{i0} x^{-1}, \\ xy_i x = y_i xy_i \ (i=1, \dots, u) \end{array} \right\rangle \\ &= \left\langle x, a_{10}, a_{11}, \dots, a_{u0}, a_{u1} \mid \begin{array}{l} x a_{i0} x^{-1} = a_{i1}, \ x a_{i1} x^{-1} = a_{i0}^{-1} a_{i1} \\ (i=1, \dots, u) \end{array} \right\rangle. \end{aligned}$$

So, the commutator subgroup $G'_u = [G_u, G_u]$ of G_u is a free group of rank $2u$; $G'_u = \langle a_{10}, a_{11}, \dots, a_{u0}, a_{u1} \mid - \rangle$. Let G''_u be $[G'_u, G'_u]$. Then the quotient group G'_u/G''_u is a free abelian group of rank $2u$.

Let $B \cong [-1, 1] \times D^2$ be an oriented 3-cell in R^4 which spans S_u as a 1-handle. Then we may assume that the attaching 0-sphere $\{-1\} \times \{0\} \cup \{1\} \times \{0\}$, 0 the center of D^2 , is contained in the left-side square of $S_u \cap R^3[0]$, as illustrated in Fig.2, by sliding B on S_u if necessary. We may also assume that the core $[-1, 1] \times \{0\}$ is contained in $R^3[0]$ by deforming B in $R^4 - S_u$. Since we may assume that B is thin sufficiently, by a slight modification of B , there is a band (=2-cell) B_0 in $R^3[0]$ so that $B = B_0[-1 \leq t \leq 1]$, as shown in Fig.2.

When we add such a 1-handle B to S_u , then the fundamental group $G_u^* = \pi_1(R^4 - h^1(S_u; B))$ has a presentation

$$G_u^* = \left\langle x, a_{10}, a_{11}, \dots, a_{u0}, a_{u1} \mid \begin{array}{l} x a_{i0} x^{-1} = a_{i1}, \quad x a_{i1} x^{-1} = a_{i0}^{-1} a_{i1}, \\ \quad (i=1, \dots, u) \\ w x w^{-1} x^{-1} = 1 \end{array} \right\rangle,$$

where w is a word of $a_{10}, a_{11}, \dots, a_{u0}, a_{u1}$; $w \in G'_u$.

In G'_u/G''_u , w is represented by $\prod_{i=1}^u a_{i0}^{m_i} a_{i1}^{n_i}$. So, in G_u/G''_u we have :

$$\begin{aligned} R = w x w^{-1} x^{-1} &= \left(\prod_{i=1}^u a_{i0}^{m_i} a_{i1}^{n_i} \right) x \left(\prod_{i=1}^u a_{i0}^{m_i} a_{i1}^{n_i} \right)^{-1} x^{-1} \\ &= \left(\prod_{i=1}^u a_{i0}^{m_i} a_{i1}^{n_i} \right) \left(\prod_{i=1}^u (x a_{i0} x^{-1})^{-m_i} (x a_{i1} x^{-1})^{-n_i} \right) \\ &= \left(\prod_{i=1}^u a_{i0}^{m_i} a_{i1}^{n_i} \right) \left(\prod_{i=1}^u a_{i1}^{-m_i} a_{i0}^{-n_i} \right) \\ &= \prod_{i=1}^u a_{i0}^{m_i + n_i} a_{i1}^{-m_i}. \end{aligned}$$

Now, we consider the normal closure of the element R in G_u/G''_u .

$$\begin{aligned}
xRx^{-1} &= x \left(\prod_{i=1}^u a_{i0}^{m_i+n_i} a_{i1}^{-m_i} \right) x^{-1} \\
&= \prod_{i=1}^u (xa_{i0}^{m_i+n_i} x^{-1}) (xa_{i1}^{-m_i} x^{-1}) \\
&= \prod_{i=1}^u a_{i1}^{m_i+n_i} \cdot a_{i0}^{m_i} a_{i1}^{-m_i} \\
&= \prod_{i=1}^u a_{i0}^{m_i} a_{i1}^{n_i} = w, \\
x^2 Rx^{-2} &= x \cdot xRx^{-1} \cdot x^{-1} = xwx^{-1} = wR^{-1}.
\end{aligned}$$

Thus, the normal closure of R in G_u/G_u'' is generated at most two elements

w and $R = xwx^{-1}x^{-1}$; that is, for a word $xwx^{-1}x^{-1}$, $w = \prod_{i=1}^u a_{i0}^{m_i} a_{i1}^{n_i}$, we

have at most two independent elements in G_u'/G_u'' . Since G_u'/G_u'' is a free abelian group of rank $2u$, for $G_u^*/G_u^{*'} = (G_u^*/G_u^{*''})/(G_u^{*'}/G_u^{*''})$ to be infinite cyclic, we need at least u relations of type $xwx^{-1}x^{-1} = 1$; that is, for S_u to be unknotted, we need at least u 1-handles that span S_u . This shows that $u(S_u) \geq u$, and completing the proof.

3.5. Corollary. *For arbitrary non-negative integers n and u , there exists a surface F in R^4 (or S^4) with $g(F) = n$ and $Kg(F) = u$.*

Proof. Since the 2-sphere S_1 given in Fig.1 is the same knot type as the spun 2-knot of the trefoil, it is easy to see that $Kg(S_1) = 1$. Now we conclude that $Kg(S_u) = u$ by Proposition 3.3.

Since the 2-sphere S_u given in the proof of Theorem 3.4 are not prime for $u \geq 2$, the following questions come to mind :

3.6. Question. Let n and u be arbitrary non-negative integers. Does there exist a prime surface F in R^4 with $g(F) = n$ and $u(F) = u$?

We have some group theoretical candidates, (see Maeda [6, Example 3.6]), but this is still open.

3.7. Question. Suppose that $[F] = [F_1] \# [F_2]$. Does it hold that

$$u[F] = u[F_1] + u[F_2] \quad ?$$

The following example shows that the answer to this 3.7 is negative.

3.8. Example. Let F_1 and F_2 be surfaces of genus 1 in R^4 as shown in Fig.3 and Fig.4, respectively.

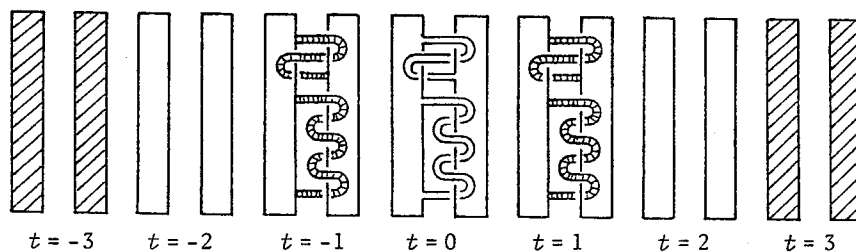


Fig. 3 : F_1 in R^4

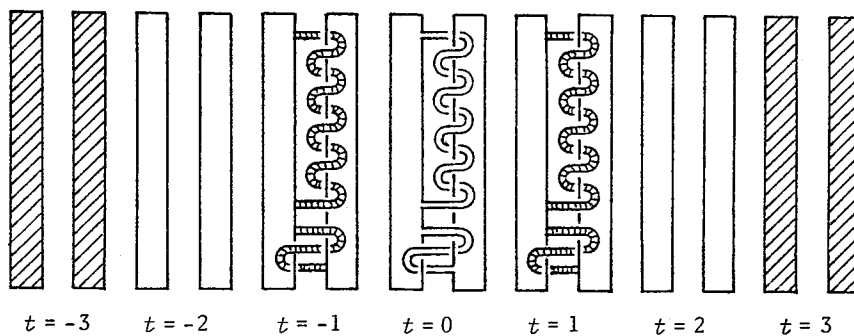


Fig. 4 : F_2 in R^4

The fundamental groups $G_1 = \pi_1(R^4 - F_1)$ and $G_2 = \pi_1(R^4 - F_2)$ have the following presentations, respectively :

$$G_1 = \langle x, y \mid xyx = yxy, xy^3 = y^3x \rangle,$$

$$G_2 = \langle x, z \mid xzx = zxx, xz^5 = z^5x \rangle.$$

Since G_1 and G_2 are groups of 3-twist-spun and 5-twist-spun 2-knots of the trefoil, respectively, both are not infinite cyclic, see Zeeman [12]. So F_1 and F_2 are knotted, and we can easily check that $u(F_1) = u(F_2) = 1$. We show that for the composition $F = F_1 \# F_2$, $u(F) = 1$. Consider Fig.5, which shows the surface $F = F_1 \# F_2$. Let B be a 1-handle, as shown in

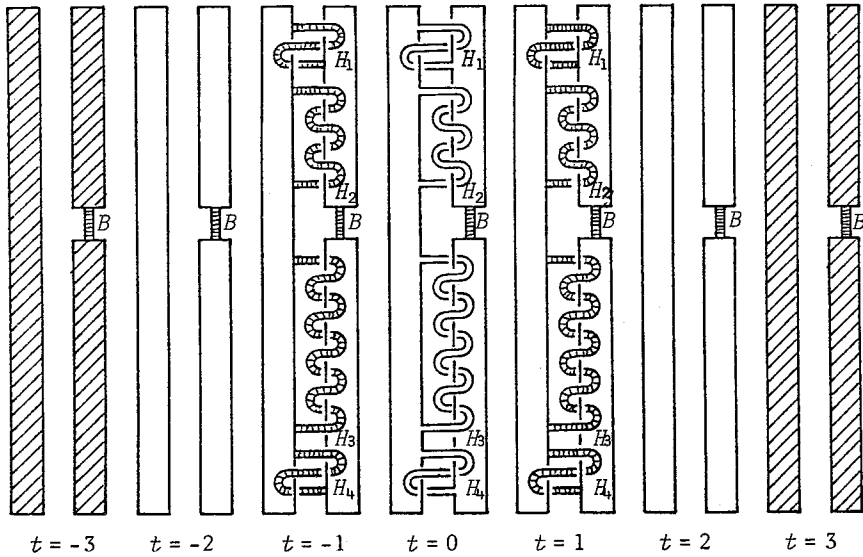


Fig. 5 : $F = F_1 \# F_2$ in R^4 , and a 1-handle B

Fig.5. By sliding on $h^1(F; B)$ the handle-parts H_1, H_2, H_3 and H_4 of F as like 1-handles, and by changing crossings of H_1, H_2, H_3 and H_4 , we have another representative F' of $[h^1(F; B)]$, as shown in Fig.6. (In Fig.6 and the following Fig.7, we give only a cross-section with $R^3[0].$) We deform F' isotopically in R^4 , and obtain another representative F''

Fig. 6 : $R^3[0] \cap F'$ Fig. 7 : $R^3[0] \cap F''$

of $[h^1(F; B)]$, as shown in Fig.7. Now, it is easily checked that F'' is an unknotted surface of genus 3.

In Example 3.8, G_1 and G_2 are groups of 2-knots, but F_1 and F_2 are surfaces of genus 1. We note that :

3.9. Question. Let F_1 and F_2 be 2-spheres in R^4 , and we suppose that $[F] = [F_1] \# [F_2]$. Does it hold that $u[F] = u[F_1] + u[F_2]$?

3.10. Question. Does there exist a surface F in R^4 with $u[F] \neq Kg[F]$?

The surface F given in Example 3.8 is one of candidates, but we can not prove that $Kg[F] \neq 1$.

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F. Hosokawa and S. Suzuki
 Department of Mathematics
 Kobe University
 Nada-ku, Kobe, 657
 Japan

T. Maeda
 Department of Mathematics
 Osaka University
 Toyonaka, Osaka, 560
 Japan