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On Convergence of Certain Iteration

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In a previous paper [1], we have shown that a number of fixed point theorems of contraction type, existence theorems of some simultaneous equations and a surjectivity theorem can be derived from a basic lemma. In a recent paper [2], Lee has derived analogously from a basic theorem various fixed point theorems. These basic results are similar in an aspect, though there are considerable differences between them. The purpose of this note is to show that we can unify them by virtue of a simple result.

Let Φ and Ψ be nonnegative real valued functions on the set $\mathbb{R}_+$ of all nonnegative real numbers. Recall that Φ is said to be Ψ -summable if for each $\xi \in \mathbb{R}_+$, the sequence $\{\Psi^n(\xi)\}_{n \in \omega}$ converges to 0 and the sequence $\{\sum_{i=0}^n (\Phi \circ \Psi^i)(\xi)\}_{n \in \omega}$ is convergent¹⁾ (cf. [1]).

Theorem. *Let X be a set, and d a nonnegative real valued function on $X \times X$, φ a nonnegative real valued function on X , and let f be a self-mapping on X . Suppose there exist nondecreasing nonnegative real valued functions Φ and Ψ on $\mathbb{R}_+$ such that Φ is Ψ -summable and*

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¹⁾ We denote by ω the set of all nonnegative integers.

$$(1) \quad d(f(x), x) \leq \Phi \circ \mathfrak{P}(x)$$

$$(2) \quad \mathfrak{P} \circ f(x) \leq \Psi \circ \mathfrak{P}(x)$$

for all x in X . Then for each $x_0 \in X$, the sequence $\{\mathfrak{P}(f^n(x_0))\}_{n \in \omega}$ converges to 0 and $\sum_{i=0}^{\infty} d(f^{i+1}(x_0), f^i(x_0)) < \infty$; hence if $\rightarrow$ is a structure on X such that $(X, \rightarrow)$ is a d -complete L -space, then $\{f^n(x_0)\}_{n \in \omega}$ is convergent.

Proof. If

$$(3) \quad \mathfrak{P} \circ f^n(x_0) \leq \Psi^n \circ \mathfrak{P}(x_0)$$

then by (2)

$$\mathfrak{P} \circ f^{n+1}(x_0) = \mathfrak{P} \circ f(f^n(x_0)) \leq \Psi \circ \mathfrak{P}(f^n(x_0)) \leq \Psi^{n+1} \circ \mathfrak{P}(x_0).$$

This, along with (2), shows that the inequality (3) holds for every $n \in \omega$.

Hence $\mathfrak{P}(f^n(x_0)) \rightarrow 0$ as $n \rightarrow \infty$. On the other hand, it follows from (1) and (3) that

$$d(f^{n+1}(x_0), f^n(x_0)) \leq \Phi \circ \mathfrak{P}(f^n(x_0)) \leq \Phi \circ \Psi^n \circ \mathfrak{P}(x_0).$$

Therefore we have $\sum_{i=0}^{\infty} d(f^{i+1}(x_0), f^i(x_0)) < \infty$.

The following corollary is the main part of the basic theorem in [2].

Corollary 1. Let S be a mapping of a metric space (X, d) into a nonempty set Y , e a nonnegative real valued function on $S(X)$, α a bounded nonnegative real valued function on $S(X)$, and β a monotone increasing function on $\mathbb{R}_+$ into $[0, 1]$. Suppose that for each $x \in X$ there exists a Δx in X satisfying the following inequalities:

$$(4) \quad d(x, \Delta x) \leq \alpha(Sx)e(Sx).$$

$$(5) \quad e(S\Delta x) \leq \beta(e(Sx))e(Sx).$$

Then for each $x_0 \in X$ there exists a Cauchy sequence $\{x_n\}_{n \in \omega}$ in X such that $e(Sx_n) \rightarrow 0$ as $n \rightarrow \infty$. If (X, d) is nonempty complete and $e \circ S$ is

continuous on X , then $e(Sa) = 0$ for some $a \in X$.

Proof. It suffices to prove the first part. Define the functions Φ and Ψ on $\mathbb{R}_+$ by $\Phi(\xi) = M\xi$ and $\Psi(\xi) = \beta(\xi)\xi$ respectively, where $\xi \in \mathbb{R}_+$ and $M = \sup \alpha \circ S(X)$. Let $\xi \in \mathbb{R}_+$. Then $\Psi^n(\xi) \leq \beta(\xi)^n \xi$ implies $\Psi^{n+1}(\xi) = \beta(\Psi^n(\xi))\Psi^n(\xi) \leq \beta(\beta(\xi)^n \xi) \cdot \beta(\xi)^n \xi \leq \beta(\xi) \cdot \beta(\xi)^n \xi$. This shows that $\Psi^n(\xi) \leq \beta(\xi)^n \xi$ holds for every $n \in \omega$. Hence $\Psi^n(\xi) \rightarrow 0$ as $n \rightarrow \infty$. Moreover

$$\sum_{i=0}^n (\Phi \circ \Psi^i)(\xi) \leq \sum_{i=0}^n \Phi(\beta(\xi)^i \xi) = M\xi \sum_{i=0}^n \beta(\xi)^i.$$

Therefore Φ is Ψ -summable. Put $\mathfrak{P} = e \circ S$. For each $x \in X$, the set $A(x)$ of all $z \in X$ such that $d(z, x) \leq \Phi \circ \mathfrak{P}(x)$ and $\mathfrak{P}(z) \leq \Psi \circ \mathfrak{P}(x)$ is nonempty since $\Delta x \in A(x)$ in view of (4) and (5). Let f be a choice function for the family $\{A(x) \mid x \in X\}$. Since f satisfies (1) and (2), our theorem ensures that $e(Sx_n) = \mathfrak{P}(x_n) \rightarrow 0$ as $n \rightarrow \infty$ and

$$(6) \quad \sum_{i=0}^{\infty} d(x_{n+1}, x_n) < \infty,$$

where $x_n = f^n(x_0)$ for each $n \in \omega$. From (6) it follows that $\{x_n\}_{n \in \omega}$ is a Cauchy sequence.

We shall now show that our theorem immediately yields an L-space version of the lemma in [1] which is fundamental in [1]. To this end, we need some definitions. By a *multivalued mapping* of a set X into another Y , we mean a mapping of X into the set of all nonempty subsets of Y . Let d be a nonnegative real valued function $X \times X$. If A and B are nonempty subsets of X then $d(A, B)$ denotes the infimum of the set of all $d(x, y)$ whenever $x \in A$ and $y \in B$. The lemma in [1] is a special case of the following:

Corollary 2. *Let $(X, \rightarrow)$ be an L-space which is d -complete for a nonnegative real valued function d on X , Y a set, d' a nonnegative real valued function on $Y \times Y$, and let F, G be multivalued mappings of X into*

Y. Suppose there exist nondecreasing nonnegative real valued functions Φ and Ψ on $\mathbb{R}_+$ such that Φ is Ψ -summable and such that the following inequalities hold over X for some selfmapping f on X :

$$d(f(x), x) \leq \Phi(d'(F(x), G(x)));$$

$$d'(F(f(x)), G(f(x))) \leq \Psi(d'(F(x), G(x))).$$

Then for each $x_0 \in X$ the sequence $\{f^n(x_0)\}_{n \in \omega}$ is convergent and

$$d'(F(f^n(x_0)), G(f^n(x_0))) \rightarrow 0$$

as $n \rightarrow \infty$.

Proof. Apply the theorem to the function φ on X defined by $\varphi(x) = d'(F(x), G(x))$ for all $x \in X$.

References

- [1] S. Kasahara: Fixed point theorems and some abstract equations in metric spaces, Math. Japonicae, 21 (1976), 165-178.
- [2] C.-M. Lee: An existence theorem via an intuitive idea and fixed point theorems, Comment. Math. Univ. Carolin., 19 (1978), 715-725.