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Lattice Ordered Autometrized Algebras III

By B. V. SUBBA RAO

In [4], which is a continuation of [3], we have studied the geometry of a representable autometrized algebra, by imposing two more natural conditions (1) $(a * (a \cup b)) \cap (b * (a \cup b)) = 0$ and (2) $a * (a \cap b) + a \cap b = a$, on any such algebra.

In §1 of this paper, we observe that the lattice of all congruence relations on a representable autometrized algebra $A = (A, +, \leq, *)$ satisfying " $a * (a \cap b) + a \cap b = a$ for all a, b in A " is a distributive lattice and any two congruence relations on A are permutable. This result generalizes the corresponding results for commutative DR ℓ -semigroups, commutative ℓ -groups, Boolean algebras and Brouwerian algebras, as the class of representable autometrized algebras satisfying " $a * (a \cap b) + a \cap b = a$ for all a, b " is wider than the class of commutative DR ℓ -semigroups. We also obtain a Chinese remainder theorem for this class of representable autometrized algebras. In §2 we introduce the notions of a direct family and a direct limit of a direct family of representable autometrized algebras and prove that the class of representable autometrized algebras is closed under the formation of direct limits.

In this paper, we freely use the definitions, results and notation of

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[3] and [4].

§ 1. We recall the definition of a representable autometrized algebra from [3].

Definition 1. A lattice ordered autometrized algebra $A = (A, +, \leq, *)$ is called a *representable autometrized algebra* if and only if

(1.1) A is a semiregular autometrized algebra,

(1.2) For every a in A , all the mappings $x \mapsto a + x$, $x \mapsto a \cup x$,

$x \mapsto a \cap x$ and $x \mapsto a * x$ are contractions (a mapping $f: A \rightarrow A$ is called a contraction mapping with respect to a metric $*$ on A if and only if $f(x) * f(y) \leq x * y$ for all x, y in A).

Throughout this section, $A = (A, +, \leq, *)$ stands for a representable autometrized algebra satisfying the condition

(1.3) $a * (a \cap b) + a \cap b = a$ for all a, b in A .

We introduce the following

Definition 2. A ternary operation f on a lattice ordered autometrized algebra $B = (B, +, \leq, *)$ is said to be a *Malcev operation* if and only if $f(x, y, y) = x$ and $f(x, x, z) = z$.

We, now, prove the following

Theorem 1. Let $A = (A, +, \leq, *)$ be a representable autometrized algebra. Then the ternary operation f defined by $f(x, y, z) = (x * (x \cap y) + x \cap z) \cup (z * (z \cap y) + z \cap x)$ is a Malcev operation on A .

Proof. $f(x, y, y) = (x * (x \cap y) + x \cap y) \cup (y * (y \cap y) + y \cap x) = x \cup (y \cap x)$
(by (1.3)) $= x$ and $f(x, x, z) = (x * (x \cap x) + x \cap z) \cup (z * (z \cap x) + z \cap x) =$

$(x \cap z) \cup z$ (by (1.3)) $= z$. Hence f is a Malcev operation on A .

Remark 1. We also have $f(x, y, z) = [x * (x \cap y) + z] \cap [z * (z \cap y) + x]$.

Proof. Let $g(x, y, z) = [x * (x \cap y) + z] \cap [z * (z \cap y) + x]$. Now,
 $g(x, y, z) = [x * (x \cap y) + z] \cap [z * (z \cap y) + x] \leq [(x * (x \cap y)) \cup (z * (z \cap y)) + z] \cap [(z * (z \cap y)) \cup (x * (x \cap y)) + x] = (x * (x \cap y)) \cup (z * (z \cap y)) + z \cap x = [x * (x \cap y) + z \cap x] \cup [z * (z \cap y) + z \cap x] = f(x, y, z)$. Further, $z * (z \cap y) + z \cap x = [z * (z \cap y) + z \cap x] \cap [z * (z \cap y) + z \cap x] \leq [z * (z \cap y) + x] \cap [z * (z \cap y) + z \cap x \cap y + (z \cap x) * (z \cap x \cap y)]$ (by (1.4)) $\leq [z * (z \cap y) + x] \cap [z * (z \cap y) + z \cap y + x * (x \cap y)]$ (by (1.2)) $= [z * (z \cap y) + x] \cap [z + x * (x \cap y)]$ (by (1.4)) $= g(x, y, z)$. Interchanging x and z , we get $x * (x \cap y) + x \cap z \leq g(z, y, x) = g(x, y, z)$. Hence, $f(x, y, z) = [z * (z \cap y) + z \cap x] \cup [x * (x \cap y) + x \cap z] \leq g(x, y, z)$. Hence $f(x, y, z) = g(x, y, z)$.

Remark 2. The above theorem is valid with a weaker hypothesis. In fact, if $A = (A, +, \leq, *)$ is a lattice ordered autometrized algebra which satisfies (1.3), then f is a Malcev operation on A . Remark 1 is valid, even when $A = (A, +, \leq, *)$ is a lattice ordered autometrized algebra which satisfies (1.3) and which is such that, for each $a \in A$, the mapping $x \mapsto a \cap x$ is a contraction.

The congruence relations on commutative DR ℓ -semigroups, Boolean algebras, Brouwerian algebras and commutative ℓ -groups are all permutable. In fact, the same is true for representable autometrized algebras, as the following theorem shows.

Theorem 2. Any two congruence relations on a representable autometrized algebra $A = (A, +, \leq, *)$ are permutable.

Proof. Let θ and φ be congruence relations on A . Assume that $x \equiv z(\theta \circ \varphi)$. Then there exists $y \in A$ such that $x \equiv y(\varphi)$ and $y \equiv z(\theta)$. Put $t = f(x, y, z) + (x * (x \cap y) + x \cap z) \cup (z * (z \cap y) + z \cap x)$. By Theorem 1, $t = f(x, y, z)$ is a Malcev operation on A . Since θ and φ satisfy the substitution property for the operations $+$, $*$, $\cup$ and $\cap$ (see Definition 2.6 and Remark 2.6.1 in [3]), we have, $x = f(x, y, y) \equiv f(x, y, z)(\theta)$ and $f(x, y, z) \equiv f(x, x, z)(\varphi) = z(\varphi)$. Thus $x \equiv t(\theta)$ and $t \equiv z(\varphi)$. Hence $x \equiv z(\varphi \circ \theta)$. Hence $\theta \circ \varphi \leq \varphi \circ \theta$. Interchanging θ and φ , we get $\varphi \circ \theta \leq \theta \circ \varphi$. Hence $\theta \circ \varphi = \varphi \circ \theta$.

Corollary 1. *Any two congruence relations on a commutative DRL-semigroup are permutable.*

We have observed (see Remark 2.4.2, [3]) that every representable autometrized algebra is a seminormal autometrized algebra ([2]).

We need the following

Lemma 1 (Prabhakara Rao, N. [2]). *The lattice of ideals of a semiregular seminormal lattice ordered autometrized algebra $A = (A, +, \leq, *)$ is a distributive lattice.*

Now, we have the following

Theorem 3. *The lattice of ideals of a representable autometrized algebra $A = (A, +, \leq, *)$ is a distributive lattice.*

Proof. Since A is a semiregular seminormal lattice ordered autometrized algebra, it follows from Lemma 1, that the lattice of ideals of A is a distributive lattice.

Theorem 4. *The structure lattice (the lattice of all congruence relations) of A is a distributive lattice.*

Proof. From Theorem 2.7 in [3] and the above Theorem 3.

We state the following theorem due to Subrahmanyam, N. V. ([5]).

Lemma 2 (Chinese Remainder theorem ([5])). *Let B be an abstract algebra, $\theta_1, \theta_2, \dots, \theta_n$ be congruence relations on B , $x_1, x_2, \dots, x_n \in B$ such that $(x_i, x_j) \in \theta_i \cup \theta_j$ for every i, j and assume further that $\theta_1, \theta_2, \dots, \theta_n$ generate a distributive and permutable sublattice of the structure lattice of B . Then, there exists $x \in B$ such that $(x, x_i) \in \theta_i$ for $i = 1, 2, \dots, n$ and such that x is unique modulo $\bigwedge_{i=1}^n \theta_i$ and hence is unique if $\bigwedge_{i=1}^n \theta_i = \omega$ (the diagonal of A).*

As any congruence relation on a representable autometrized algebra is a congruence relation in the sense of universal algebra and as the structure lattice of a representable autometrized algebra is distributive and any two congruence relations on it are permutable, from Subrahmanyam's Chinese remainder theorem, we immediately have the following Chinese remainder theorem for representable autometrized algebras. Theorem 2.4 and Lemma 2 yield the following.

Theorem 5 (Chinese remainder theorem). *Let $A = (A, +, \leq, *)$ be a representable algebra. Let $\theta_1, \theta_2, \dots, \theta_n$ be congruence relations on A and $x_1, x_2, \dots, x_n \in A$ such that $(x_i, x_j) \in \theta_i \cup \theta_j$ for every i and j . Then, there exists $x \in A$ such that $(x, x_i) \in \theta_i$ for $i = 1, 2, \dots, n$; also, such x is unique modulo $\bigwedge_{i=1}^n \theta_i$ and hence is unique if $\bigwedge_{i=1}^n \theta_i = \omega$*

(the diagonal of A).

Corollary 2. Let $\theta_1, \theta_2, \dots, \theta_n$ be congruence relations on A ; $x_1, x_2, \dots, x_n \in A$ such that $\theta_i \circ \theta_j = 1$ ($= A \times A$) for all $i, j, i \neq j$. Then, there exists x in A such that $(x, x_i) \in \theta_i$ for $i = 1, 2, \dots, n$.

The following is a consequence of Theorem 2.7 [3] and Theorem 5 above.

Corollary 3. Let $I_1, I_2, \dots, I_n$ be ideals of a representable autometrized algebra $A = (A, +, \leq, *)$ $x_1, x_2, \dots, x_n \in A$ and assume that $x_i \equiv x_j \pmod{(I_i \vee I_j)}$ for all i, j . Then, there exists $x \in A$ such that $x \equiv x_i \pmod{I_i}$ for $i = 1, 2, \dots, n$.

Now, we have the following well known result for ℓ -groups (Corollary 3, P. 305, [1]).

Corollary 4 (T. Nakano). Let $N_1, N_2, \dots, N_r$ be ℓ -ideals of a commutative ℓ -group G , and let $a_1, a_2, \dots, a_r$ be elements of G such that $a_i \equiv a_j \pmod{N_i + N_j}$ for every pair i, j . Then, there exists an element $a \in G$ such that $a \equiv a_i \pmod{N_i}$ for $i = 1, 2, \dots, r$.

§ 2. We begin with the following

Definition 3. Let $S = (S, +, \leq, *)$ and $T = (T, +, \leq, *)$ be two lattice ordered autometrized algebras. We say that a mapping $f: S \rightarrow T$ is a homomorphism if and only if f satisfies

$$(H1) \quad f(a + b) = f(a) + f(b) \quad \text{for all } a, b \text{ in } S,$$

$$(H2) \quad f(a * b) = f(a) * f(b) \quad \text{for all } a, b \text{ in } S,$$

$$(H3) \quad f(a \cup b) = f(a) \cup f(b) \quad \text{for all } a, b \text{ in } S,$$

$$(H4) \quad f(a \cap b) = f(a) \cap f(b) \quad \text{for all } a, b \text{ in } S.$$

A homomorphism f from a lattice ordered autometrized algebra S into a lattice ordered autometrized algebra T is called an *isomorphism* if and only if $f: S \rightarrow T$ is bijection and $f^{-1}: T \rightarrow S$ is a homomorphism.

Throughout this section, we consider only representable autometrized algebras. We do not assume that any such algebra satisfies the condition: $a * (a \cap b) + a \cap b = a$ for all a, b .

Definition 4. By a *direct family* A of representable autometrized algebras, we mean a triple, $A = (A_i, f_{ij}, I)$ of the following objects:

- (i) A directed (up directed) set $(I, \leq)$
- (ii) Representable autometrized algebras $A_i = (A_i, +, \leq, *)$, $i \in I$.
- (iii) Homomorphisms $f_{ij}: A_i \rightarrow A_j$ for all $i \leq j$, such that $f_{jk}f_{ij} = f_{ik}$ if $i \leq j \leq k$ and $f_{ii}: A_i \rightarrow A_i$ is the identity mapping for all $i \in I$.

Definition 5. Given a direct family $A = (A_i, f_{ij}, I)$ of representation autometrized algebras, by a direct limit of this family A , we mean a representable autometrized algebra $B = (B, +, \leq, *)$ together with homomorphisms $f_i: A_i \rightarrow B$, $i \in I$ such that the diagram

$$\begin{array}{ccc} A_i & \xrightarrow{f_i} & B \\ & \searrow f_{ij} & \nearrow f_j \\ & A_j & \end{array}$$

is commutative whenever $i \leq j$, such that, if $C = (C, +, \leq, *)$ is a representable autometrized algebra and $g_i: A_i \rightarrow C$ are homomorphisms such that the diagram

$$\begin{array}{ccc} A_i & \xrightarrow{g_i} & C \\ & \searrow f_{ij} & \nearrow g_j \\ & A_j & \end{array}$$

is commutative whenever $i \leq j$, then there exists a unique homomorphism $h: B \rightarrow C$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{h} & C \\ f_i \swarrow & & \nearrow g_i \\ & A_i & \end{array}$$

is commutative for each $i \in I$.

Now, we have the following

Theorem 6. *Every direct family of representable autometrized algebras has a direct limit which is unique up to isomorphism.*

Proof. Let $A = (A_i, f_{ij}, I)$ be a direct family of representable autometrized algebras. Let $K = \bigcup_{i \in I} A_i$ be the disjoint union of A_i , $i \in I$.

(We assume that A_i are pairwise disjoint; if this is not the case, we have to form the disjoint union, say $\bigcup \{A_i \times \{i\} \mid i \in I\}$). Define $\equiv$ on K by $x \equiv y$ if and only if there exists $k \geq i, j$ such that $f_{ik}(x) = f_{jk}(y)$ where $x \in A_i$ and $y \in A_j$. If $x \in K$, then $x \in A_i$ for some $i \in I$ and $f_{ii}(x) = f_{ii}(x)$. Hence $x \equiv x$ for all $x \in K$, so that $\equiv$ is reflexive. $\equiv$ is clearly symmetric. Now let $x \equiv y$, and $y \equiv z$, $x \in A_i$, $y \in A_j$, $z \in A_k$. Then there exists $\ell \in I$ such that $\ell \geq i, j$ and $f_{i\ell}(x) = f_{j\ell}(y)$; and there exists $m \in I$ such that $m \geq j, k$ and $f_{jm}(y) = f_{km}(z)$. Since I is directed, there exists $t \in I$ such that $t \geq \ell, m$. Then, $t \geq \ell \geq i$ and $t \geq m \geq k$ and $f_{it}(x) = f_{\ell t} f_{i\ell}(x) = f_{\ell t} f_{j\ell}(y) = f_{jt} f_{jm}(y) = f_{kt} f_{km}(z) = f_{kt}(z)$. This implies $x \equiv z$; so that $\equiv$ is transitive. Hence $\equiv$ is an equivalence relation in K . Let $A = K/\equiv$ be the quotient set (the set of all equivalence classes). Now, we make A into a representable autometrized algebra. Let $\bar{x}, \bar{y} \in A$. Let $x \in A_i$ and

$y \in A_j$ for some $i, j \in I$. Let $k \geq i$ and $k \geq j$. Put $x' = f_{ik}(x)$ and $y' = f_{jk}(y)$. Then define $\bar{x} + \bar{y} = \overline{x' + y'}$, $\bar{x} \cup \bar{y} = \overline{x' \cup y'}$, $\bar{x} \cap \bar{y} = \overline{x' \cap y'}$ and $\bar{x} * \bar{y} = \overline{x' * y'}$. First, we show that the operations $+$, $\cup$, $\cap$ and $*$ are well defined on A . For this, let $\bar{x} = \bar{x}_1$ and $\bar{y} = \bar{y}_1$ and $x_1 \in A_\ell$, $y_1 \in A_m$. Let $n \geq \ell$ and $n \geq m$. Put $x'_1 = f_{\ell n}(x_1)$, $y'_1 = f_{mn}(y_1)$. To show that $+$ is well defined, we have to show that $\overline{x' + y'} = \overline{x'_1 + y'_1}$. Since $\bar{x} = \bar{x}_1$, there exists an element $p \in I$ such that, $p \geq i, \ell$ and $f_{ip}(x) = f_{\ell p}(x_1)$. Since $\bar{y} = \bar{y}_1$, there exists an element $q \in I$ such that $q \geq j, m$ and $f_{jq}(y) = f_{mq}(y_1)$. Let $t \in I$ such that $t \geq k, t \geq p, t \geq q$ and $t \geq n$. Then we have $f_{kt}(x' + y') = f_{kt}(x') + f_{kt}(y') = f_{kt}f_{ik}(x) + f_{kt}f_{jk}(y) = f_{it}(x) + f_{jt}(y) = f_{pt}f_{ip}(x) + f_{qt}f_{jq}(y) = f_{pt}f_{\ell p}(x_1) + f_{qt}f_{mq}(y_1) = f_{\ell t}(x_1) + f_{mt}(y_1) = f_{nt}f_{\ell n}(x_1) + f_{nt}f_{mn}(y_1) = f_{nt}(x'_1) + f_{nt}(y'_1) = f_{nt}(x'_1 + y'_1)$. This implies $x' + y' \equiv x'_1 + y'_1$ so that $\overline{x' + y'} = \overline{x'_1 + y'_1}$. Hence $+$ is well defined. In a similar manner we can show that $\cup$, $\cap$ and $*$ are well defined. Now, let $\bar{x}, \bar{y}, \bar{z} \in A$ and let $x \in A_i$, $y \in A_j$ and $z \in A_k$. Let $t \geq i, t \geq j$ and $t \geq k$. Put $x' = f_{it}(x)$, $y' = f_{jt}(y)$ and $z' = f_{kt}(z)$. Now, $\bar{x} + (\bar{y} + \bar{z}) = \bar{x} + \overline{y' + z'} = \overline{x' + (y' + z')} = \overline{(x' + y') + z'} = \overline{x' + y'} + \bar{z} = (\bar{x} + \bar{y}) + \bar{z}$. This implies that $+$ is associative. Also $\bar{x} + \bar{y} = \overline{x' + y'} = \overline{y' + x'} = \bar{y} + \bar{x}$ so that $+$ is commutative. Let O_i be the identity element of A_i and O_j be the identity element of A_j . Let $k \geq i$ and $k \geq j$. Then $f_{ik}(O_i) = O_k = f_{jk}(O_j)$, so that $O_i \equiv O_j$. Hence $\bar{O}_i = \bar{O}_j$ for all $i, j \in I$. So, let us write $\bar{O} = \bar{O}_i$ for all $i \in I$. Let $\bar{x} \in A$ and $x \in A_i$ for some $i \in I$. Then $\bar{x} + \bar{O} = \overline{x + O_i} = \bar{x}$. This implies $\bar{O}$ is the identity element of A . Thus $A = (A, +)$ is a commutative semigroup with identity $\bar{O}$.

Now, let $\bar{x}, \bar{y}, \bar{z} \in A$ and let $x \in A_i$, $y \in A_j$ and $z \in A_k$ for some $i, j, k \in I$.

$k \in I$. Let $t \geq i, j, k$. Put $x' = f_{it}(x)$, $y' = f_{jt}(y)$ and $z' = f_{kt}(z)$. Then $\bar{x} \cup \bar{x} = \overline{x \cup x} = \bar{x}$, $\bar{x} \cup \bar{y} = \overline{x' \cup y'} = \overline{y' \cup x'} = \bar{y} \cup \bar{x}$ and $(\bar{x} \cup \bar{y}) \cup \bar{z} = \overline{x' \cup y' \cup z'} = \overline{(x' \cup y') \cup z'} = \overline{x' \cup (y' \cup z')} = \bar{x} \cup \overline{y' \cup z'} = \bar{x} \cup (\bar{y} \cup \bar{z})$. Similarly we can prove that $\bar{x} \cap \bar{x} = \bar{x}$, $\bar{x} \cap \bar{y} = \bar{y} \cap \bar{x}$ and $(\bar{x} \cap \bar{y}) \cap \bar{z} = \bar{x} \cap (\bar{y} \cap \bar{z})$. Further, $\bar{x} \cup (\bar{x} \cap \bar{y}) = \bar{x} \cup \overline{x' \cap y'} = \overline{x' \cup (x' \cap y')} = \overline{x'} = \bar{x}$ (since $f_{it}(x) = f_{tt}(x')$). Similarly we can show that $\bar{x} \cap (\bar{x} \cup \bar{y}) = \bar{x}$. Hence $(A, \cup, \cap)$ is a lattice. Also, $\bar{x} + (\bar{y} \cup \bar{z}) = \bar{x} + \overline{y' \cup z'} = \overline{x' + (y' \cup z')} = \overline{(x' + y') \cup (x' + z')} = \overline{x' + y' \cup x' + z'} = (\bar{x} + \bar{y}) \cup (\bar{x} + \bar{z})$, and $\bar{x} + (\bar{y} \cap \bar{z}) = \bar{x} + \overline{y' \cap z'} = \overline{x' + (y' \cap z')} = \overline{(x' + y') \cap (x' + z')} = \overline{x' + y' \cap x' + z'} = (\bar{x} + \bar{y}) \cap (\bar{x} + \bar{z})$. Hence $(A, +, \leq)$ is a lattice ordered semigroup with identity 0 .

Let $\bar{x}, \bar{y} \in A$ and $x \in A_i, y \in A_j$ for some $i, j \in I$. Let $t \geq i, j$. Put $x' = f_{it}(x)$ and $y' = f_{jt}(y)$. Now, $\bar{x} * \bar{y} = \overline{x' * y'} = \overline{(x' * y') \cup \bar{0}_t}$ (since $x' * y' \geq \bar{0}_t$ in A_t) $= \overline{x' * y' \cup \bar{0}} = (\bar{x} * \bar{y}) \cup \bar{0}$. This implies $\bar{x} * \bar{y} \geq \bar{0}$. Also $\bar{x} * \bar{y} = \bar{0}$ implies that $\overline{x' * y'} = \bar{0}_t$, which implies that there exists $\ell \geq t$ such that $f_{t\ell}(x' * y') = f_{t\ell}(\bar{0}_t)$. This implies $f_{t\ell}(x') * f_{t\ell}(y') = \bar{0}_\ell$ so that $f_{t\ell}(x') = f_{t\ell}(y')$. Hence $x' \equiv y'$. But $x \equiv x'$ and $y \equiv y'$ (since $f_{it}(x) = f_{\ell t}(x')$ and $f_{jt}(y) = f_{\ell t}(y')$). So, $x \equiv y$ and hence $\bar{x} = \bar{y}$. Thus $\bar{x} * \bar{y} = \bar{0}$ implies $\bar{x} = \bar{y}$. Clearly $\bar{x} * \bar{x} = \overline{x * x} = \bar{0}_i = \bar{0}$. Further $\bar{x} * \bar{y} = \overline{x' * y'} = \overline{y' * x'} = \bar{y} * \bar{x}$. Now, let $\bar{x}, \bar{y}, \bar{z} \in A$ and let $x \in A_i, y \in A_j$ and $z \in A_k$. Let $t \geq i, j, k$. Put $x' = f_{it}(x)$, $y' = f_{jt}(y)$ and $z' = f_{kt}(z)$. Then, we have $x' * z' \leq x' * y' + y' * z'$. This implies $(x' * z') \cup (x' * y' + y' * z') = (x' * y' + y' * z')$, so that we have $\overline{x' * z' \cup x' * y' + y' * z'} = \overline{x' * y' + y' * z'}$. That is, $(\bar{x} * \bar{z}) \cup (\bar{x} * \bar{y} + \bar{y} * \bar{z}) = \bar{x} * \bar{y} + \bar{y} * \bar{z}$. Hence $\bar{x} * \bar{z} \leq \bar{x} * \bar{y} + \bar{y} * \bar{z}$. Thus, $*$ is a metric operation on A . Hence $A = (A, +, \leq, *)$ is a lattice ordered

autometrized algebra. Now, let $\bar{x} \geq \bar{0}$ and $x \in A_i$. Then, we have $\bar{x} \cup \bar{0} = \bar{x}$, i.e., $\overline{x \cup 0_i} = \bar{x}$. So, there exists $j \geq i$ such that $f_{ij}(x \cup 0_i) = f_{ij}(x)$. That is, $f_{ij}(x) \cup f_{ij}(0_i) = f_{ij}(x)$ i.e., $f_{ij}(x) \cup 0_j = f_{ij}(x)$. Hence $0_j \leq f_{ij}(x)$. Hence $f_{ij}(x) * 0_j = f_{ij}(x)$ (by semiregularity of A_j). Hence $\bar{x} * \bar{0} = \overline{f_{ij}(x) * 0_j} = \overline{f_{ij}(x)} = \bar{x}$ (since $f_{jj}f_{ij}(x) = f_{ij}(x)$). Thus $\bar{x} \geq \bar{0}$ implies $\bar{x} * \bar{0} = \bar{x}$. Hence $A = (A, +, \leq, *)$ is a semiregular lattice ordered autometrized algebra.

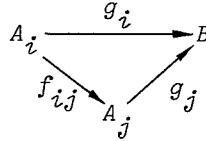
Let $\bar{a} \in A$ and let $a \in A_n$ for some $n \in I$. We now show that the mapping $\bar{x} \mapsto \bar{a} + \bar{x}$ is a contraction. Let $\bar{x}, \bar{y} \in A$ and let $x \in A_i, y \in A_j$ for some $i, j \in I$. Let $k \geq i, j, n$. Put $a' = f_{nk}(a)$, $x' = f_{ik}(x)$ and $y' = f_{jk}(y)$. Now, we have $(\overline{a' + x' * a' + y'}) \cup \overline{x' * y'} = \overline{((a' + x') * (a' + y')) \cup (x' * y')} = \overline{x' * y'}$ (since $x' \mapsto a' + x'$ is a contraction in A_k). This implies $\overline{a' + x' * a' + y'} \leq \overline{x' * y'}$. That is, $(\bar{a} + \bar{x}) * (\bar{a} + \bar{y}) \leq \bar{x} * \bar{y}$. Hence the mapping $\bar{x} \mapsto \bar{a} + \bar{x}$ is a contraction. Similarly, we can show that the mappings $\bar{x} \mapsto \bar{a} \cup \bar{x}$, $\bar{x} \mapsto \bar{a} \cap \bar{x}$ and $\bar{x} \mapsto \bar{a} * \bar{x}$ are all contractions. Hence $A = (A, +, \leq, *)$ is a representable autometrized algebra.

For each $i \in I$, define the natural map $f_i: A_i \rightarrow A$ by $f_i(x) = \bar{x}$. Then $f_i(x + y) = \overline{x + y} = \bar{x} + \bar{y} = f_i(x) + f_i(y)$, $f_i(x \cup y) = \overline{x \cup y} = \bar{x} \cup \bar{y} = f_i(x) \cup f_i(y)$, $f_i(x \cap y) = \overline{x \cap y} = \bar{x} \cap \bar{y} = f_i(x) \cap f_i(y)$ and $f_i(x * y) = \overline{x * y} = \bar{x} * \bar{y} = f_i(x) * f_i(y)$, for all $x, y \in A_i$. Hence f_i is a homomorphism. Now, let $i, j \in I$ such that $i \leq j$. Let $x \in A_i$. Then, $f_i(x) = \bar{x} = \overline{f_{ij}(x)}$ (since $f_{ij}(x) = f_{jj}f_{ij}(x)$) = $f_jf_{ij}(x)$. This implies that the diagram

$$\begin{array}{ccc} A_i & \xrightarrow{f_i} & A \\ & \searrow f_{ij} & \nearrow f_j \\ & A_j & \end{array}$$

is commutative, whenever $i \leq j$.

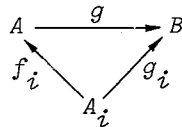
Now, let $B = (B, +, \leq, *)$ be a representable autometrized algebra and $g_i: A_i \rightarrow B$ be homomorphisms such that the diagram



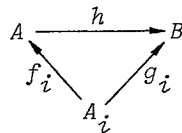
is commutative, whenever $i \leq j$.

Then, define $g: A \rightarrow B$ by $g(\bar{x}) = g_i(x)$ if $x \in A_i$. (Note that each element of A is of the form $\bar{x}$, where x is in some A_i , $i \in I$). We show that g is a well defined mapping. Let $\bar{x} = \bar{y}$ and $x \in A_i$, $y \in A_j$ for some $i, j \in I$. Then there exists $k \in I$ such that $k \geq i, j$ and $f_{ik}(x) = f_{jk}(y)$. This implies $g_k f_{ik}(x) = g_k f_{jk}(y)$, which again implies (by the commutativity of the above diagram) that $g_i(x) = g_j(y)$. Hence $g(\bar{x}) = g(\bar{y})$. Thus g is well defined. Let $\bar{x}, \bar{y} \in A$ and $x \in A_i$, $y \in A_j$ for some $i, j \in I$. Then there exists $k \in I$ such that $k \geq i, j$. Put $x' = f_{ik}(x)$ and $y' = f_{jk}(y)$. Then, $g(\bar{x} + \bar{y}) = g(\overline{x' + y'}) = g_k(x' + y') = g_k(x') + g_k(y') = g(\overline{x'}) + g(\overline{y'}) = g(\bar{x}) + g(\bar{y})$ (since $f_{ik}(x) = x' = f_{kk}(x')$ and $f_{jk}(y) = y' = f_{kk}(y')$). Similarly we can show that $g(\bar{x} \cup \bar{y}) = g(\bar{x}) \cup g(\bar{y})$, $g(\bar{x} \cap \bar{y}) = g(\bar{x}) \cap g(\bar{y})$ and $g(\bar{x} * \bar{y}) = g(\bar{x}) * g(\bar{y})$. Hence g is a homomorphism.

Let $i \in I$ and $x \in A_i$. Then $g f_i(x) = g f_i(x) = g(\bar{x}) = g_i(x)$. This implies that $g(f_i) = g_i$ for each $i \in I$. Thus g is a homomorphism such that the diagram

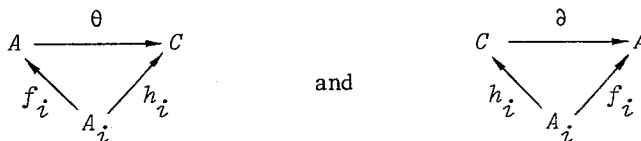


is commutative, for every $i \in I$. We, now, show that g is unique with respect to this property. Assume that there is a homomorphism $h: A \rightarrow B$ such that the diagram



is commutative, for every $i \in I$. Let $\bar{x} \in A$ and $x \in A_i$ for some $i \in I$. Then, $g(x) = g_i(x) = hf_i(x) = h(x)$. This implies that $g = h$. Hence g is a unique homomorphism with the above property. Hence $A = (A, +, \leq, *)$, together with the natural homomorphisms $f_i: A_i \rightarrow A$, $i \in I$, is a direct limit of the direct family $A = (A_i, f_{ij}, I)$ of representable autometrized algebras.

To show that this direct limit A of A_i is unique up to isomorphism, assume that $C = (C, +, \leq, *)$ is a representable autometrized algebra such that C , together with homomorphisms $h_i: A_i \rightarrow C$, $i \in I$, is a direct limit of the direct family $A = (A_i, f_{ij}, I)$. Then, since A and C are direct limits of the family A , there exist unique homomorphisms $\theta: A \rightarrow C$ and $\partial: C \rightarrow A$ such that each of the diagrams



is commutative, for each $i \in I$. This implies that $\partial\theta: A \rightarrow A$ is a homomorphism such that the diagram

$$\begin{array}{ccc}
 A & \xrightarrow{\partial\theta} & A \\
 f_i \swarrow & & \nearrow f_i \\
 & A_i &
 \end{array}$$

is commutative, for each $i \in I$, since $\partial\theta f_i(x) = \partial f_i \theta(x) = \partial h_i(x) = f_i(x)$ for every $x \in A_i$. But, there is already the identity homomorphism

$I_A : A \rightarrow A$ such that the diagram

$$\begin{array}{ccc}
 A & \xrightarrow{I_A} & A \\
 f_i \swarrow & & \nearrow f_i \\
 & A_i &
 \end{array}$$

is commutative for each $i \in I$. Hence, by the uniqueness of such homomorphism, it follows that $\partial\theta = I_A$. Similarly, we can show that $\partial\theta = I_C$, the identity homomorphism on C . Hence $\theta : A \rightarrow C$ is an isomorphism. Hence the direct limit A of A is unique upto isomorphism. This completes the proof.

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