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Characters of Commutative Semigroups II — Topological Semigroups

By James A. ANDERSON

The purpose of this paper is to describe, up to isomorphism, the character semigroup of a commutative topological semigroup. Descriptions of the character semigroup up to isomorphism in special cases have been given by Fulp [3] and Bowman [2].

All semigroups in this paper are commutative with identity. Hom (S, T) will denote the set of all continuous homomorphisms from the topological semigroup S to the topological semigroup T.

Definition 1.1. Let S be a commutative topological semigroup. A *character* χ of S is a continuous mapping of S into C, not identically 0, such that $\chi(s) = 1$ or 0 for all $s \in S$, and $\chi(s + t) = \chi(s)\chi(t)$.

Definition 1.2. Let S^* be the set of all characters of S. For χ, ψ in S^*, define $\chi \circ \psi$ by $(\chi \circ \psi)(s) = \chi(s)\psi(s)$. $(S^*, \circ)$ is called the *character semigroup* of S.

Definition 1.3. Let S be a commutative topological semigroup, and H^0 a fixed Hausdorff topological group with 0 such that 0 is an isolated

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point. $\mathrm{Hom}(S, G^0)$, with the 0-map excluded, is called the *generalized* H^0 *character semigroup* of S and is denoted by $S^{\#}$. An element of $S^{\#}$ is called a *generalized* H^0 *character* or simply a *generalized character.*

The following lemma may be found in Hewitt and Ross [4]:

Lemma 1.4. *Let* S *be a group (semigroup),* T *a topological group (semigroup), and* $f: S \to T$ *a homomorphism. Let* T *be the topology on* S *induced by* f. S *with the topology* T *is a topological group (semigroup).*

Lemma 1.5. *Let* (S, T_i) *be a topological semigroup (group) for each topology* T_i *on* S. *Let* T *be the topology on* S *with subbase* $\bigcup_i \mathsf{T}_i$. (S, T) *is a topological semigroup (group).*

Proof. The proof is straightforward.

Definition 1.6. Let $(G_\alpha, g_{\alpha'}^{\alpha}, A)$ be the corresponding triple for a union of groups S. Let T be a topology on G. G is a union of topological groups if:

(1) G is a topological semigroup,

(2) G_α with the relative topology is a topological group.

Proposition 1.7. *Let* S *be a topological semigroup. There exists a union of topological groups* G *such that* $G^{\#} = S^{\#}$.

Proof. Let $G = S \otimes \mathbf{Z}$, and $\eta: S \to G$ be the canonical homomorphism. For any union of topological groups H and continuous homomorphism $f: S \to H$, there exists a homomorphism $\overline{f}_H: G \to H$ such that $\overline{f}_H \eta = f$. Let $F = \{\overline{f}_H \mid f: S \to H$ is a continuous homomorphism into a union of groups $H\}$.

Let T be the topology on G induced by the functions in F. It follows from Lemma 1.5 and 1.6 that G with the topology T is a union of topological groups. From the construction, $G^{\#} = S^{\#}$.

Proposition 1.8. *Let S be a topological semigroup. There exists a Hausdorff union of groups K such that $S^{\#} = K^{\#}$.*

Proof. Let $G = S \otimes Z$ as above. Let $F = \{\overline{f}_H \mid f : S \to H$ is a homomorphism into a Hausdorff union of topological groups $H\}$. Let $K = G/\sim$ where $\sim$ is the congruence defined by $g \sim g'$ if $\overline{f}(g) = \overline{f}(g')$ for all $\overline{f} \in F$. Every $\overline{f}_H \in F$ factors through K. Hence there exists $\hat{f}_H$ such that

$$\begin{array}{ccc} G & \longrightarrow & K \\ & \searrow^{\overline{f}_H} & \downarrow \hat{f}_H \\ & & H \end{array}$$

commutes. Let $\hat{F} = \{\hat{f}_H \mid \overline{f}_H \in F\}$. Let K have the topology induced by the functions in $\hat{F}$. This is the desired union of topological groups.

Definition 1.9. K as developed in Proposition 1.8 is called the *separation of G by unions of groups.*

Lemma 1.10. *Let G be a union of topological groups with triple $(G_\alpha, g^{\alpha}_{\alpha'}, A)$. Let $\chi \in G^{\#}$ and $\Delta_\chi = \{\alpha \mid \alpha \in A$ and $\chi(\alpha) = 1_G\}$. Δ_χ is a filter, and $\bigcup_{\alpha \in \Delta} G_\alpha$ is an open and closed subsemigroup of G.*

Proof. The proof is straightforward.

Definition 1.11. Let χ and Δ_χ be defined as in Lemma 1.9. Δ_χ is called the filter *associated* with χ. If Δ_χ contains a least element, χ is called a *principal character.*

Definition 1.12. If $\bigcup_{\alpha\in\Delta} G_\alpha$ is open and closed, call Δ a *c-filter*.

Lemma 1.13. *Let G be a union of topological groups with triple $(G_\alpha, g^\alpha_{\alpha'}, A)$. Let Δ be a c-filter of A. If Δ has a least element β, then $G^\#_\beta$ is isomorphic to the group K_Δ of all elements of $G^\#$ associated with the filter Δ, where G_β has the relative topology.*

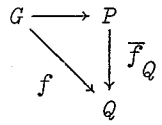
Proof. If $\chi \in K_\Delta$, then $\chi|G_\beta$ is a character of G_β. Conversely, given $\chi_\beta \in G^\#_\beta$, define χ by

$$\chi(g_\alpha) = \begin{cases} \chi_\beta(g_\alpha 1_\beta) & \text{if} \quad \alpha \geq \beta \\ 0 \quad \text{otherwise} \end{cases}$$

$\chi \to \chi_\beta$ is the desired isomorphism.

Proposition 1.14. *Let $(G_\alpha, g^\alpha_{\alpha'}, A)$ be the triple for a union of groups G. G has a greatest group image P in the category of topological semigroups.*

Proof. Let P be the greatest algebraic group image of G. For every group Q and continuous homomorphism $f: G \to P$, there exists $\overline{f}_Q : P \to Q$ such that



commutes.

Let $F = \{\overline{f}_Q \mid f$ is a continuous homomorphism from G into a topological group $Q\}$. Let P have the topology T induced by the homomorphisms in F. P with the topology T is the desired group.

Proposition 1.15. *Let* $(G_\alpha, g_{\alpha'}^{\alpha}, A)$ *be the triple for a union of groups* G. G *has a greatest group image* K *in the category of Hausdorff topological semigroups.*

Proof. Let K be the separation by groups of P in the previous proposition. K is the desired topological group.

Proposition 1.16. *Let* S *be the algebraic direct sum of* $\{G_\alpha \mid \alpha \in A\}$. *Let* P *be the greatest group image of* G. *There exists* φ *such that*

$$\begin{array}{ccc} G & \longrightarrow & S \\ & \searrow & \downarrow \varphi \\ & & P \end{array}$$

commutes. Let S *have the topology induced by* φ, *and* $J = \ker \varphi$. $S/\overline{J}$ *is the greatest group image in the topology of Hausdorff topological semigroups.*

Proof. Let $\eta : S \to Q$ be a continuous homomorphism of S into a Hausdorff topological group Q. $J \subset \ker \eta$. Therefore $\overline{J} \subset \ker \eta$. Therefore η factors through $S/\overline{J}$ and the result is immediate.

Lemma 1.17. *Let* G *be a union of topological groups with triple* $(G_\alpha, g_{\alpha'}^{\alpha}, A)$. *Let* Δ *be a c-filter of* A *and* $S_\Delta = \{\chi \mid \chi \in G^{\#}$ *and has* Δ *as its associated filter*$\}$. S_Δ *is isomorphic with* $T_\Delta^{\#}$ *where* T_Δ *is the greatest Hausdorff group image of* $\bigcup_{\alpha \in \Delta} G_\alpha$.

Proof. The proof follows immediately from the definition of greatest group image.

Let M and L be topological groups. Let φ be a continuous homo-

morphism of M into L. For each $f \in \mathrm{Hom}(L, H^0)$, define a mapping $\varphi^{\#}(f)$ of M into H by $\varphi^{\#}(f(a)) = f\varphi(a)$. $\varphi^{\#}(f) \in \mathrm{Hom}(M, H^0)$. Hence $\varphi^{\#}$ is a homomorphism of $L^{\#}$ into $M^{\#}$.

Definition 1.18. $\varphi^{\#}$ is called the H^0 *adjoint* of the homomorphism φ. If f is a character then $\varphi^{\#}$ is simply called the *adjoint* of φ and is denoted by $\varphi^{\#}$.

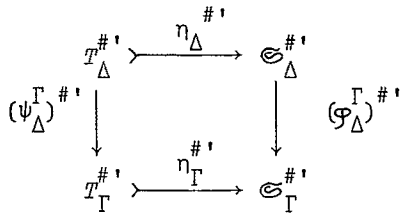
From Lemma 1.16 it follows that $G^{\#}$ is a union of groups, $\bigcup_{\Delta \in \overline{F}} T_{\Delta}^{\#}$ where $\overline{F}$ is the set of c-filters of A. Let $\Delta, \Gamma \in \overline{F}$, $\Gamma \subset \Delta$. Let $\theta_{\Gamma} : \bigcup_{\alpha \in \Gamma} G_{\alpha} \to T_{\Gamma}$ be the continuous homomorphism of $\bigcup_{\alpha \in \Gamma} G_{\alpha}$ onto its greatest Hausdorff group image. θ_{Δ} is a continuous homomorphism of $\bigcup_{\alpha \in \Gamma} G_{\alpha}$ into T_{Δ}. This induces a continuous homomorphism $\psi_{\Delta}^{\Gamma} : T_{\Gamma} \to T_{\Delta}$.

Theorem 1.19. *$G^{\#}$ is a union of groups with triple $(T_{\Delta}^{\#}, \psi_{\Delta}^{\Gamma^{\#}}, \overline{F})$ where $\overline{F}$ is considered as a semilattice under intersection.*

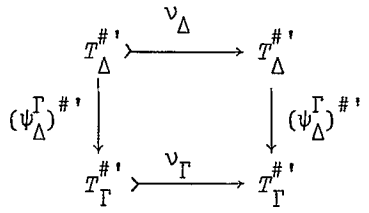
Proof. Let $\mathfrak{S}_{\Gamma} = \varinjlim_{\Delta} \{G_{\alpha} \mid \alpha \in \Gamma\}$ and $\varphi_{\Gamma}^{\Delta}$ the induced homomorphism between $\mathfrak{S}_{\Delta}$ and $\mathfrak{S}_{\Gamma}$ as given in [1]. From the construction of T_{Δ}, we have the following commuting diagram of groups and homomorphisms:

$$\begin{array}{ccccc}
\bigcup_{\alpha \in \Gamma} G_{\alpha} & \twoheadrightarrow & \mathfrak{S}_{\Gamma} & \twoheadrightarrow^{\eta_{\Gamma}} & T_{\Gamma} \\
\downarrow & & \downarrow \varphi_{\Delta}^{\Gamma} & & \downarrow \psi_{\Delta}^{\Gamma} \\
\bigcup_{\alpha \in \Delta} G_{\alpha} & \twoheadrightarrow & \mathfrak{S}_{\Delta} & \twoheadrightarrow_{\eta_{\Delta}} & T_{\Delta}
\end{array}$$

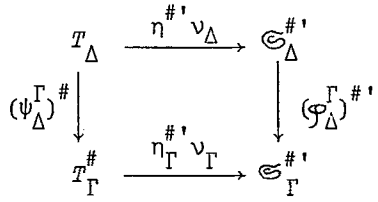
Therefore



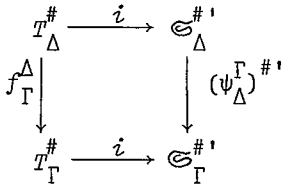
commutes, where $S^{\#'}$ is the set of all homomorphisms between the groups S and H, and $\varphi^{\#'}$ is the algebraic H^0 adjoint of φ as defined in [1]. But



commutes where ν_Δ and ν_Γ are natural embeddings. Hence



commutes. $G^{\#} \subset G^{\#'}$. Hence



commutes where $(T_\Delta, f_\Gamma^\Delta, \overline{F})$ is the triple for $G^{\#}$, and $i : G^{\#} \to G^{\#'}$ is the natural embedding. Let V be the union of groups with triple $(T^{\#}, (\psi_\Delta^\Gamma)^{\#}, \overline{F})$, and W the union of groups with triple $(T^{\#'}, (\psi_\Delta^\Gamma)^{\#''}, \overline{F})$.

Define $\nu : V \to W$ By $\omega(t_\Delta) = \nu_\Delta(t_\Delta)$ for all $t_\Delta \in T_\Delta^{\#}$ and define $\omega: W \to G^{\#'}$ by $\omega(t'_\Delta) = \eta_\Delta^{\#'}(t'_\Delta)$ for $t'_\Delta \in T_\Delta^{\#'}$. ν and ω are monomorphisms. Therefore $f_\Gamma^\Delta = (\psi_\Delta^\Gamma)^{\#}$.

Corollary. G^* is a union of troups with triple $(T_\Delta^*, \psi_\Delta^{\Gamma*}, \overline{F})$.

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