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# Characters of Commutative Semigroups III — Bicharacters

By James A. ANDERSON

In this paper, a description is given of the bicharacter semigroups of pairs of algebraic semigroups, and pairs of topological semigroups.

All semigroups in this paper are commutative with identity. Descriptions of bicharacter semigroups are given up to isomorphism.

Definition 1.1. A *bicharacter* of semigroups $S$ and $T$ is a map from $S \times T$ into C, not identically 0, such that $|\chi(s, t)| = 0$ or 1,

$\chi(s, t + t') = \chi(s, t)\chi(s, t')$, and $\chi(s + s', t) = \chi(s, t)\chi(s', t)$ for all $s, s' \in S$ and for all $t, t' \in T$.

Definition 1.2. The *bicharacter semigroup* $(S, T)^+$ of semigroup $S$ and $T$ is the set of bicharacters of $S$ and $T$ together with the multiplication $\circ$ defined by $(\eta \circ \chi)(s, t) = \eta(s, t)\chi(s, t)$.

Definition 1.3. Let $S$ and $T$ be semigroups, and $H^0$ a fixed group with 0. Bihom$(S, T; H^0)$ without the 0 map is called *the generalized $H^0$ bicharacter semigroup of $S$ and $T$* and is denoted $(S, T)\#\#$. An element of $(S, T)\#\#$ is called a *generalized $H^0$ bicharacter* or simply a *gener-*

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*alized bicharacter* of $S$ and $T$.

**Proposition 1.4.** $(S, T)\#\# \cong (S \otimes T)\#$ *where* $(S \otimes T)\#$ *is the generalized* $H^0$ *character semigroup* [1] *of* $S \otimes T$.

*Proof.* The proof follows directly from the definition of tensor product.

**Proposition 1.5.** $(S, T)\#\# \cong (S \otimes Z, T \otimes Z)\#\#$ *where* $Z$ *is the additive group of integers.*

*Proof.* $(S, T)\#\# \cong (S \otimes T)\# \cong (S \otimes T \otimes Z)\# \cong (S \otimes Z \otimes T \otimes Z)\# \cong (S \otimes Z \otimes T \otimes Z)\#\#$.

Since $S \otimes Z$ and $T \otimes Z$ are unions of groups, in the remainder of this section we shall consider only bicharacters of unions of groups.

**Definition 1.6.** Let $A$ and $B$ be semilattices and $C = A \times B$. $F$ is a *bifilter* of $C$ if the following conditions are satisfied:

(1) If $(x, y) \in F$, $(a, b) \in C$ and $(a, b) \geq (x, y)$ then $(a, b) \in F$.

(2) If $(a, b) \in F$ and $(a, b') \in F$, then $(a, b + b') \in F$.

(3) If $(a, b) \in F$ and $(a', b) \in F$, then $(a + a', b) \in F$.

$\psi(A, B)$ will denote the lattice of bifilters of $C$. $(A, B)$ will denote the semilattice of finitely generated bifilters of $C$.

Lemma 1.7 is due to N. Kimura [5]. Also see [3]. Lemma 1.8 is due to Anderson and Kimura [4]:

**Lemma 1.7.** *Let* $A$ *and* $B$ *be semilattices. Then* $A \otimes B \cong (A, B)$.

**Lemma 1.8.** *Let* $(G_\alpha, g^\alpha_{\alpha'}, A)$ *and* $(H_\beta, h^\beta_{\beta'}, B)$ *be corresponding tri-*

*ples for unions of groups* $S$ *and* $T$ *respectively. For* $\Delta \in (A, B)$ *let* $\mathfrak{S}_\Delta = \{G_\alpha \otimes H_\beta, g^\alpha_{\alpha'} \otimes h^\beta_{\beta'} \mid (\alpha, \beta), (\alpha', \beta') \in \Delta\}$. *Let* $T_\Delta$ *be the colimit of* $\mathfrak{S}_\Delta$. *For* $\Delta \subset \Gamma$ *let* $\sigma^\Delta_\Gamma$ *be the induced homomorphisms between* $T_\Delta$ *and* $T_\Gamma$. *Then* $(T_\Delta, \sigma^\Gamma_\Delta, \varphi(A, B))$ *is the corresponding triple for* $S \otimes T$.

Let $\varphi\#$ denote the $H^0$ adjoint of the homomorphism $\varphi$ as described in [1]. Lemma 1.9 may be found in [2] and is given here without proof:

Lemma 1.9. *Let* $U$ *be a union of groups with corresponding triple* $(U_\delta, \psi^\delta_{\delta'}, D)$. *Let* $F_\wedge$ *be the set of filters of* $D$ *considered as a semilattice with intersection as the operation, and* $F_\vee$ *the same set with the dual lattice operation of* $F_\wedge$. *For* $f \in F$, *let* $P_f = \{U_\delta, \psi^\delta_{\delta'} \mid \delta, \delta' \in f\}$. *Let* $K_f = \operatorname{colim} P_f$, *and* $\omega^f_{f'}$ *the induced map between* $K_f$ *and* $K_{f'}$. *Then* $(K^\#_f, (\omega^f_{f'})\#, F_\wedge)$ *is the corresponding triple of* $U\#$.

Letting $U = S \otimes T$ in Lemma 1.9 we have:

Lemma 1.10. $(S \times T)^{++}$ *has corresponding triple* $(K^\#_f, (\omega^f_{f'})\#, F_\wedge)$ *where* $K_f = \operatorname{colim}\{T_\Delta, \sigma^\Gamma_\Delta \mid \Delta, \Gamma \in f\}$, $T_\Delta = \operatorname{colim}\{G_\alpha \otimes H_\beta, g^\alpha_{\alpha'} \otimes h^\beta_{\beta'} \mid (\alpha, \beta), (\alpha', \beta') \in \Delta\}$, *and* $F_\wedge$ *is the set of filters of* $\varphi(A, B)$.

Proposition 1.11. *Let* $\psi_\vee(A, B)$ *be considered as a semilattice under the operation* $\vee$ *where* $\Delta \vee \Gamma$ *is the bifilter generated by* $\Delta \cup \Gamma$. *Then* $\psi_\vee(A, B) \cong F_\vee$.

*Proof.* Let $f \in F_\vee$, then $B_f = \bigcup_{\Delta \in f} \Delta$ is a bifilter. Conversely, let $B$ be a bifilter and $f$ be the set of all finitely generated bifilters contained in $B$, then $f$ is a filter of $(A, B)$ and $B = \bigcup_{\Delta \in f} \Delta$. Hence

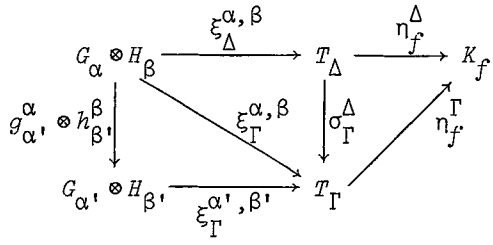
$f \mapsto B_f$ is a one to one correspondence between $F_\nu$ and $\psi_\nu(A, B)$. Further since $f \le f'$ iff $B_f \le B_{f'}$, it is also an isomorphism.

Given $\Delta \in \psi(A, B)$. Let $W_\Delta = \operatorname{colim}\{G_\alpha \otimes H_\beta, g^\alpha_{\alpha'} \otimes h^\beta_{\beta'} \mid (\alpha, \beta), (\alpha', \beta') \in \Delta\}$. For $\Delta \subset \Gamma$, there is an induced homomorphism $\gamma^\Delta_\Gamma : W_\Delta \to W_\Gamma$ (for details see similar results in [4] for finitely generated bifilters).

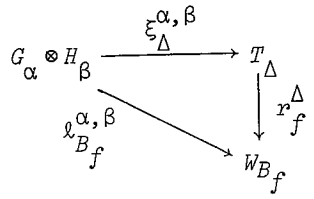
Theorem 1.12. *$(W^\#_\Delta, \gamma^{\Delta^\#}_\Gamma, \psi_\wedge(A, B))$ is a corresponding triple for $(S \times T)\#\#$.*

*Proof.* First we show that $K_f \cong W_{B_f}$. Let $\{\ell^{\alpha,\beta}_{B_f} : G_\alpha \otimes H_\beta \to W_{B_f}\}$, $\{\xi^{\alpha,\beta}_\Delta : G_\alpha \otimes H_\beta \to T_\Delta\}$ and $\{\eta^\Delta_f : T_\Delta \to K_f\}$ be colim cones for $W_{B_f}$, $T_\Delta$ and $K_f$ respectively. Consider $\{\eta^\Delta_f \xi^{\alpha,\beta}_\Delta : G_\alpha \otimes H_\beta \to K_f\}$. Let $\Delta \subset \Gamma$, $(\alpha, \beta) \ge (\alpha', \beta')$, $(\alpha, \beta) \in \Delta$, and $(\alpha', \beta') \in \Gamma$. For $g_\alpha \otimes h_\beta \in G_\alpha \otimes H_\beta$,

$$\eta^\Delta_f \xi^{\alpha,\beta}_\Delta(g_\alpha \otimes h_\beta) = \eta^\Gamma_f \sigma^\Delta_\Gamma \xi^{\alpha,\beta}_\Delta(g_\alpha \otimes h_\beta) = \eta^\Gamma_f \xi^{\alpha,\beta}_\Gamma(g_\alpha \otimes h_\beta)$$
$$= \eta^\Gamma_f \xi^{\alpha',\beta'}_\Gamma(g^\alpha_{\alpha'} \otimes h^\beta_{\beta'})(g_\alpha \otimes h_\beta)$$



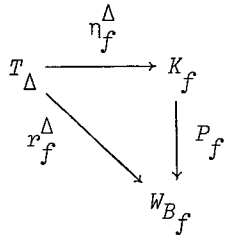

Therefore $\eta^\Delta_f \xi^{\alpha,\beta}_\Delta = \eta^\Gamma_f \xi^{\alpha',\beta'}_\Gamma(g^\alpha_{\alpha'} \otimes h^\beta_{\beta'})$. Hence by definition of colimit, there exists $q_f : W_{B_f} \to K_f$ such that $\eta^\Gamma_f \xi^{\alpha,\beta}_\Delta = q_f \ell^{\alpha,\beta}_{B_f}$. Also $\ell^{\alpha,\beta}_{B_f} = \ell^{\alpha',\beta'}_{B_f}(g^\alpha_{\alpha'} \otimes h^\beta_{\beta'})$ for $(\alpha, \beta), (\alpha', \beta') \in \Delta$ and $(\alpha, \beta) \ge (\alpha', \beta')$. By definition of colimit, there exists $r^\Delta_f : T_\Delta \to W_{B_f}$ such that



commutes. Let $\Delta, \Gamma \in f$, $\Delta \subset \Gamma$ and $t_\Delta \in T_\Delta$. Then

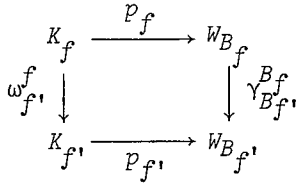
$$
\begin{aligned}
r_f^\Delta(t_\Delta) &= r_f^\Delta \xi_\Delta^{\alpha,\beta}(g_\alpha \otimes h_\beta) \quad \text{for some } g_\alpha \otimes h_\beta \text{ with } (\alpha, \beta) \in \Delta \\
&= \ell_{B_f}^{\alpha,\beta}(g_\alpha \otimes h_\beta) \\
&= \ell_f^{\alpha',\beta'}(g_{\alpha'}^\alpha \otimes h_{\alpha'}^\beta)(g_\alpha \otimes h_\beta) \quad \text{for } (\alpha', \beta') \in \Gamma,\ (\alpha, \beta) \geq (\alpha', \beta') \\
&= r_f^\Gamma \xi_\Gamma^{\alpha',\beta'}(g_{\alpha'}^\alpha \otimes h_{\beta'}^\beta)(g_\alpha^\alpha \otimes h_\beta^\beta) \\
&= r_f^\Gamma \sigma_\Gamma^\Delta \xi_\Delta^{\alpha,\beta}(g_\alpha \otimes h_\beta) \\
&= r_f^\Gamma \sigma_\Gamma^\Delta(t_\Delta).
\end{aligned}
$$

Hence $r_f^\Delta = r_f^\Gamma \sigma_\Gamma^\Delta$ and by definition of colimit, there exists $p_f : K_f \to W_{B_f}$ such that



commutes.

Using the definition of the colimit it is straightforward to show $p_f q_f$ is the identity map on $W_{B_f}$ and $q_f p_f$ is the identity map on $K_f$. Hence $W_{B_f} \cong K_f$. Next we show that



commutes where $f \subset f'$.

Let $k_f \in K_f$. Then

$$
\begin{aligned}
p_{f'}\ \omega^{f}_{f'}(k_f) &= p_{f'}\ \omega^{f}_{f'}\ \eta^{\Delta}_{f}(t_\Delta) \quad \text{for some } t_\Delta \in T_\Delta \text{ and } \Delta \in f. \\
&= p_{f'}\ \eta^{\Delta}_{f'}(t_\Delta) \\
&= p_{f'}\ \eta^{\Delta}_{f'}\ \xi^{\alpha,\beta}_{\Delta}(g_\alpha \otimes h_\beta) \quad \text{for some } g_\alpha \otimes h_\beta \in G_\alpha \otimes H_\beta \text{ and } (\alpha, \beta) \in \Delta \\
&= \ell^{\alpha,\beta}_{B_{f'}}(g_\alpha \otimes h_\beta) \\
&= \gamma^{Bf}_{B_{f'}}\ \ell^{\alpha,\beta}_{B_f}(g_\alpha \otimes h_\beta) \\
&= \gamma^{Bf}_{B_{f'}}\ p_f\ \eta^{\Delta}_{f}\ \xi^{\alpha,\beta}_{\Delta}(g_\alpha \otimes h_\beta) \\
&= \gamma^{Bf}_{B_{f'}}\ p_f\ \eta^{\Delta}_{f}(t_\Delta) \\
&= \gamma^{Bf}_{B_{f'}}\ p_f\ k_f
\end{aligned}
$$

Therefore $p_{f'}\ \omega^{f}_{f'} = \gamma^{Bf}_{B_{f'}}\ p_f$ and the above diagram commutes.

From the first two parts of the proof and Proposition 1.11, we conclude that $(K_f, \omega^{f}_{f'}, F_V)$ and $(W_\Delta, \gamma^{\Delta}_{\Gamma}, \psi_V(A, B))$ are corresponding triples for isomorphic unions of groups. The theorem is then immediate.

Corollary. $(W^*_\Delta, \gamma^{\Delta^*}_{\Gamma}, \psi_\wedge(A, B))$ *is a corresponding triple for* $(S \times T)^+$.

## 2. Topological bicharacters of commutative topological semigroups.

Definition 2.1. Let $S$ and $T$ be commutative topological semigroups and $H^0$ a fixed, Hausdorff, topological group with isolated 0. A *topological $H^0$ bicharacter* $\chi$ of $S \times T$ is a bicontinuous mapping of $S \times T$ into $H^0$, not identically 0 such that $\chi(s, t + t') = \chi(s, t)\chi(s, t')$ and $\chi(s + s', t) = \chi(s, t)\chi(s', t)$. The set of topological $H^0$ bicharacters considered as a subsemigroup of $\mathrm{Bihom}(S, T; H^0)$ is aclled the *gener-*

*alized* $H^0$ *topological bicharacter semigroup* of $S$ and $T$, and is denoted $(S \times T)^!$

Let $S$ and $T$ be commutative topological semigroups, and $S \otimes T$, the tensor product of $S$ and $T$ in the category of commutative semigroups. For each bihomomorphisms $B_u$ of $S \times T$ into a commutative topological semigroup $U$, there exists a homomorphism $C_u$ such that

$$\begin{array}{ccc} S \times T & \xrightarrow{\quad t \quad} & S \otimes T \\ & {\scriptstyle B_u} \searrow \quad \swarrow {\scriptstyle C_u} & \\ & U & \end{array}$$

commutes. Let $P = \{C_u \mid B_u : S \times T \to U$ is a bicontinuous bihomomorphism$\}$. Let $\mathcal{T}$ be the topology on $S \otimes T$ induced by the homomorphisms in $P$.

Proposition 2.3. *$S \otimes T$, with the topology $\mathcal{T}$, is the tensor product of $S$ and $T$ in the category of commutative topological semigroups. Denote this tensor product by $S \overline{\otimes} T$.*

*Proof.* The proposition follows from the construction of $\mathcal{T}$.

Proposition 2.4. *The tensor product of $S$ and $T$ exists in the category of commutative Hausdorff topological semigroups.*

*Proof.* Let $K$ be the separation of $S \otimes T$ by semigroups as described in [2]. $K$ is the desired semigroup.

Lemma 2.5. $(S \times T)^! \cong (S \overline{\otimes} T)\# \cong K\#$ *where* $(S \overline{\otimes} T)\#$ *is the generalized* $H^0$ *character semigroup of* $S \overline{\otimes} T$ *as described in* [2].

*Proof.* The lemma follows directly from the definition of tensor pro-

duct.

Definition 2.6. Let $S$ and $T$ be commutative semigroups and $\mathcal{T}$ a topology on $S \times T$ such that $S \times T$ is a topological semigroup. Let $(S \times T)\#\#$ denote the semigroup of bicontinuous bihomomorphisms from $S \times T$ into $H^0$ with the exclusion of the 0-map.

Proposition 2.7. *Let $S$ and $T$ be commutative semigroups and $\mathcal{T}$ a topology on $S \times T$. Let $G = S \otimes Z$ and $H = T \otimes Z$ where $Z$ is the additive group of integers. There exists a topology $\mathcal{U}$ on $G \times H$ such that $(S \times T)\#\# \cong (G \times H)\#\#$.*

*Proof.* Using the bicontinuous bihomomorphisms from $S \times T$ into topological semigroups, induce a topology $\mathcal{T}'$ on $S \otimes T$ as in Proposition 2.3. Let $(S \otimes' T)$ denote $S \otimes T$ with the topology $\mathcal{T}'$. Then $(S \otimes' T)\# = (S \times T)\#\#$. But $(S \otimes T) \otimes Z \cong G \otimes H$. Hence there is a topology on $G \otimes H$ so that $(G \otimes H)\# \cong (S \otimes' T)\#$. Let $t : G \times H \to G \otimes H$ be the canonical homomorphism. $t$ induces a topology on $G \times H$ so tnat $(G \times H)\#\# \cong (G \otimes H)\#$.

Corollary. *Let $S$ and $T$ be commutative topological semigroups. Let $G = S \otimes Z$ and $H = T \otimes Z$. There exists topologies on $G$ and $H$ so that $(G \times H)^! \cong (S \times T)^!$.*

Defininiton 2.8. Let $(G_\alpha, g_{\alpha'}^{\alpha}, A)$ be the corresponding triple for a union of groups $G$. Let $\mathcal{T}$ be a topology on $G$. $G$ is a *union* of topological groups if the following conditions are satisfied:

(1) $G$ is a topological semigroup.

(2) $G_\alpha$ with the relative topology is a topological group.

Proposition 2.9. *Let $S$ and $T$ be commutative topological semigroups. Let $G = S \otimes Z$ and $H = T \otimes Z$. There exists topologies on $G$, $H$, and $G \otimes H$ so that $(G \times H)^{!} \cong (S \times T)^{!} \cong (G \otimes H)\#$ and $G$, $H$, and $G \otimes H$ are unions of topological groups.*

*Proof.* Induce the topology $T'$ for $S \otimes T$ using only bicontinuous bihomomorphisms from $S \times T$ into unions of topological groups, in the proof of Proposition 2.7.

The following lemma may be found in [2]:

Lemma 2.10. *Let $G$ be a union of topological groups with triple $(G_\alpha, \varphi^{\alpha}_{\alpha'}, A)$. $G\#$ is a union of groups with corresponding triple $(S^{\#}_{\Delta}, \psi^{\Gamma\#}_{\Delta}, F_\wedge)$ where $S^{\#}_{\Delta}$ is the greatest Hausdorff group image of $\bigcup_{\alpha \in \Delta} G$, $\psi^{\Gamma}_{\Delta}$ is the induced map, and $F_\wedge$ is the semilattice of c-filters of $A$ under intersection.*

Theorem 2.11. *Let $G$ and $H$ be unions of topological groups with corresponding triples $(G_\alpha, g^{\alpha}_{\alpha'}, A)$ and $(H_\beta, h^{\beta}_{\beta'}, B)$ respectively. Let $G \otimes H$ with the topology of Proposition 2.9, have corresponding triple $(T_\Delta, \sigma^{\Gamma}_{\Delta}, \varphi(A, B))$. $(G \times H)^{!}$ has corresponding triple $(S^{\#}_{\Delta}, \psi^{\Gamma\#}_{\Delta}, F_\wedge)$ where $S^{\#}_{\Delta}$ is the greatest Hausdorff image of $\bigcup_{\Delta \in f} T_\Delta$, $\sigma^{\Gamma}_{\Delta}$ is the induced map, and $F_\wedge$ is the semilattice of c-filters of $(A, B)$.*

*Proof.* The theorem follows directly from Proposition 2.9 and Lemma 2.10.

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