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Remarks on Contrapositionally Complemented BCK-Algebras

(NB BCK-Algebras)

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H. Rasiowa introduced the notion of an abstract algebra which is called a contrapositionally complemented positive implication algebra (see [3]). This algebra is defined by the following way.

Let X be a positive implicative BCK-algebra (for detail, see [2]). If an operation $\sim\colon x \to \sim x$ in X satisfying

(1) $$\sim x * y = \sim y * x$$

for any $x, y \in X$, is given, then X is said to be *contrapositionally complemented* (simply *c. c.*) *positive implicative.*

If a BCK-algebra has the greatest element 1, it is said to be *bounded.* Let us define $\sim x$ by $1 * x$ in a bounded BCK-algebra X. Then X satisfies the equality (1)(see [1]). Therefore *any bounded BCK-algebra is c. c.*

Generally, we may consider a BCK-algebra (not necessarily positive implicative) with a non-trivial operation $\sim$ satisfying (1). Here the word 'non-trivial' means that the operation $\sim$ does not carry every element to 0. We are interested of course in non-trivial cases.

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We shall give a simple example of a BCK-algebra with non-trivial operation $\sim$:

Example 1. Let $X = \{0, 1, \cdots, n\}$ $(n \geq 1)$. We define

$$x * y = \begin{cases} x - y & \text{if } y < x, \\ 0 & \text{if } x \leq y, \end{cases}$$

and

$$\sim x = n - x.$$

Then X is a commutative BCK-algebra. To prove (1), let $\sim x \leq y$. This implies $n - x \leq y$. Hence $y \leq n - x$. This means that (1) holds for $\sim x \leq y$. Let $y < \sim x$. Then $y < n - x$, which implies $x < n - y$. Hence the both sides of (1) are equal to $n - x - y$. Therefore (1) is true in X. Moreover this algebra satisfies $\sim\sim x = x$ and $x * \sim y = y * \sim x$.

Example 2. Let X be a partially ordered set with the least element 0 such that every pair of non-zero distinct elements is incomparable. We define the operation $*$ as follows:

$$x * y = \begin{cases} 0 & \text{if } x \leq y, \\ x & \text{otherwise.} \end{cases}$$

Under this operation $*$, X is an implicative BCK-algebras (for detail see [2]). In this algebra, we can introduce many different non-trivial NB-structures. Consider a fixed element a $(\neq 0)$ of X, and define

$$\sim x = a * x.$$

Then, by a basic property of a BCK-algebra, we have

$$\sim x * y = (a * x) * y = (a * y) * x = \sim y * x.$$

Hence X satisfies the condition (1). Therefore, this structure is given by $\sim a = 0$ and $\sim x = a$ for $x \neq a$, which is non-trivial.

Definition 1. A BCK-algebra with an unary operation $\sim$ satisfying $\sim x * y = \sim y * x$ is called a *contrapositionally complemented algebra* (*c. c. algebra*) or an *NB-algebra*.

Let X be a c.c. BCK-algebra. By (1), we have

$$0 = \sim x * \sim x = \sim\sim x * x,$$

which implies

(2) $$\sim\sim x \leq x.$$

Since 0 is the least element in X, (2) implies $\sim\sim 0 = 0$. By (1) and (2) we have

$$\sim x * \sim y = \sim\sim y * x \leq y * x$$

Hence $y \leq x$ implies $\sim x \leq \sim y$, and any c.c. BCK-algebra satisfies $\sim x * \sim y \leq y * x$, i.e. *any c.c. BCK-algebra is an NBN-algebra* (see [1]).

If a c.c. BCK-algebra X satisfies $\sim\sim x = x$, then, by (2), for any x in X, $0 \leq \sim x$ implies $x = \sim\sim x \leq \sim 0$. Hence X is bounded. Therefore we have the following

Theorem 1. *If a c.c. BCK-algebra X satisfies $\sim\sim x = x$, then X is a bounded BCK-algebra. The greatest element is ~ 0.*

Example 3. Let X be a partially ordered set given in Figure 1. We define the operation $*$ on X by the following table. Then X is a commutative BCK-algebra.

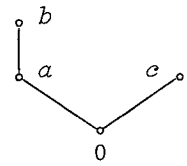


Figure 1

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let us suppose that X becomes to be a c.c. BCK-algebra by an operation $\sim$. For any $x \in X$, $0 \leq x$ implies $\sim x \leq \sim 0$. If $\sim 0 = c$, then $\sim a \leq \sim b \leq 0$ or c. Therefore we have three cases:

1) $\sim a = \sim b = 0$, 2) $\sim a = 0$, $\sim b = c$, 3) $\sim a = \sim b = c$.

We shall consider the first case. Since X is a c.c. BCK-algebra, we obtain $\sim 0 * b = \sim b * 0$. This implies $c = c * b = 0 * 0 = 0$, which is a contradiction.

Next we shall consider the second case. We have $\sim a * 0 = \sim 0 * a$, $\sim a * 0 = 0$ and $\sim 0 * a = c * a = c$. Hence this case does not occur.

Finally we shall consider the third case. Since X is a c.c. BCK-algebra, we have $\sim b * c = \sim c * b$. Hence $0 = \sim c * b$. This means $\sim c = b$ or $\sim c = 0$. If $\sim c = b$, then we have $0 = \sim 0 * c = \sim c * 0 = b * 0 = b$, which is a contradiction. Suppose $\sim c = 0$. Since $c * x = \sim 0 * x = \sim x * 0 = \sim x$, we have $c = c * x = \sim x$ for $x = a$, b, and $0 = c * c = \sim c = 0$. Hence $\sim 0 * x = \sim x * 0$. If $x = 0$ or $x = b$, the both sides of

$$c * x = \sim a * x = \sim x * a \tag{3}$$

are c. If $x = c$, the both sides of (3) are 0. Hence $\sim a * x = \sim x * a$. If $x = 0$ or $x = a$, both sides of

$$c * x = \sim b * x = \sim x * b \tag{4}$$

are c. If $x = b$, the both sides of (4) are 0. Therefore $\sim b * x = \sim x * b$. For every x, we have $\sim c * x = \sim x * c = 0$.

We conclude that X is a c.c. BCK-algebra if $\sim 0 = \sim a = \sim b = c$ and $\sim c = 0$. In this algebra, $\sim\sim a = \sim\sim b = \sim c = 0$. Hence $\sim\sim x = x$ does not hold.

H. Yutani communicated to me that there are more two non-trivial structures satisfying (1) which are obtained from another branch $\{0, a, b\}$. Consequently, we have three non-trivial structures on X.

In my Note [1], we introduced an operation N on a bounded BCK-algebra by defining $Nx = 1 * x$. Then we proved $Nx * y = Ny * x$. Hence in general

a bounded BCK-algebra has a non-trivial structure satisfying (1).

Theorem 2. *If a bounded BCK-algebra* X *is c. c. and satisfies* $\sim 0 = 1$, *then* $Nx = \sim x$ *for any* x *in* X.

Proof. Since X satisfies (1), we have

$$\sim 0 * x = \sim x * 0 = \sim x.$$

By $\sim 0 = 1$, the left side of the above equation is equal to $1 * x = Nx$. Then it follows that $Nx = \sim x$ for any x in X, which completes the proof.

Theorem 2 is an important result. This shows that the c.c. algebraic structure on a bounded BCK-algebra satisfying $\sim 0 = 1$ is uniquely determined.

As a consequence of Theorem 2, we have the following

Corollary. *If* $\sim Nx = x$ *holds for any* x *in a bounded c. c. BCK-algebra* X, *then* $Nx = \sim x$ *for every* x.

Theorem 3. *In any bounded c. c. BCK-algebra,* $\sim 1 = 0$ *holds.*

Proof. The relation (1) implies

$$\sim Nx * y = \sim y * Nx.$$

Hence letting $y = 1$, we have

$$0 = \sim 1 * Nx,$$

which implies $\sim 1 \leq Nx$ for any x. Since $N1 = 0$, we have $\sim 1 = 0$, as desired.

We shall conclude this Note by an interesting result on a c.c. BCK-algebra.

Theorem 4. *If X is a c. c. BCK-algebra, then*

$$\sim x = \sim 0 * x$$

for every x in X.

Proof. Letting $y = 0$ in

$$\sim x * y = \sim y * x,$$

we obtain

$$\sim x = \sim x * 0 = \sim 0 * x.$$

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