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PDF issue: 2026-08-11

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(Citation)
Mathematics Seminar Notes,7(3):639-646

(Issue Date)
1979

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/E0001512

(URL)
https://hdl.handle.net/20.500.14094/E0001512

MATHEMATICS SEMINAR NOTES
Vol. 7 (1979)

Domains in Grassmann Manifolds with Vanishing Cohomology Groups

By Kiyoshi WATANABE*

Introduction. It was pointed out by Serre [8] that a domain D in $\mathbf{C}^n$ is a domain of holomorphy if it satisfies the conditions $H^q(D, \mathcal{O}) = 0$ for $1 \leq q \leq n-1$, where $\mathcal{O}$ is the sheaf of germs of holomorphic functions on D. This result of Serre was improved by Laufer [6], who showed that a domain D in an n-dimensional Stein manifold is a Stein manifold if $H^q(D, \mathcal{O})$ is finite dimensional for $1 \leq q \leq n-1$. By the use of methods of Laufer and Kajiwara-Kazama [3], Mori [7] proved that a domain D in an n-dimensional Stein manifold with $H^q(D, \mathcal{O}) = 0$ for $2 \leq q \leq n-1$ is a Stein manifold if it satisfies $H^1(D, \mathcal{A}_L) = 0$ for a complex Lie group L, where $\mathcal{A}_L$ is the sheaf of germs of holomorphic mappings in L.

On the other hand, Kajiwara-Watanabe [5] showed that a proper subdomain D in the 2-dimensional complex projective space $\mathbf{P}^2$ is a Stein manifold if it satisfies $H^1(D, \mathcal{A}_L) = 0$ for a complex Lie group L. The aim of this paper is to extend these results. We shall show that a domain D in an n-dimensional Grassmann manifold is a Stein manifold with $H^2(D, \mathbf{Z}) = 0$ if

Received December 3, 1979.

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it satisfies $H^1(D, \mathcal{O}^*) = 0$ and $H^q(D, \mathcal{O}) = 0$ for $2 \le q \le n-1$, where $\mathcal{O}^*$ is the sheaf of germs of non-vanishing holomorphic functions on D, and $\mathbb{Z}$ is the constant sheaf of integers.

§ 1. Envelope of holomorphy and envelope of meromorphy. Let M be a complex manifold. If ϕ is a locally biholomorphic mapping of a connected complex manifold D in M, (D, ϕ) is called a *domain over* M. Let (D, ϕ) and (D', ϕ') be domains over M. A holomorphic mapping λ of D in D' with $\phi = \phi' \circ \lambda$ is called a *mapping of* (D, ϕ) *in* (D', ϕ'). Let f be a holomorphic (or meromorphic) function on D. A holomorphic (or meromorphic) function f' on D' with $f = f' \circ \lambda$ is called a *holomorphic* (or *meromorphic*) *continuation of* f *to* (λ, D', ϕ'). Let $\mathcal{F}$ be a family of holomorphic (or meromorphic) functions on D. If every holomorphic (or meromorphic) function of $\mathcal{F}$ has a holomorphic (or meromorphic) continuation to (λ, D', ϕ'), (λ, D', ϕ') is called a *holomorphic* (or *meromorphic*) *completion of* (D, ϕ) *with respect to* $\mathcal{F}$. A holomorphic (or meromorphic) completion $(\tilde{\lambda}, \tilde{D}, \tilde{\phi})$ of (D, ϕ) with respect to $\mathcal{F}$ is called an *envelope of holomorphy* (or *meromorphy*) *of* (D, ϕ) *with respect to* $\mathcal{F}$ if the following conditions are satisfied:

Let (λ', D', ϕ') be another holomorphic (or meromorphic) completion of (D, ϕ) with respect to $\mathcal{F}$. Then there exists a mapping ψ of (D', ϕ') in $(\tilde{D}, \tilde{\phi})$ with $\tilde{\lambda} = \psi \circ \lambda'$ such that $(\psi, \tilde{D}, \tilde{\phi})$ is a holomorphic (or meromorphic) completion of (D', ϕ') with respect to $\mathcal{F}' = \{f' : f' \circ \lambda' \in \mathcal{F}\}$.

If $\mathcal{F}$ is the family of all holomorphic (or meromorphic) functions on D, an envelope of holomorphy (or meromorphy) of (D, ϕ) with respect to

$\mathcal{F}$ is called an *envelope of holomorphy* (or *meromorphy*) *of* (D, ϕ). An envelope of holomorphy (or meromorphy) of (D, ϕ) with respect to $\{f\}$ is called a *domain of holomorphy* (or *meromorphy*) *of* f.

Proposition 1 (Kajiwara-Sakai [4]). *Let* (D, ϕ) *be a domain over a complex manifold* M *and* $\mathcal{F}$ *be a family of holomorphic* (*or meromorphic*) *functions on* D. *Then there exists uniquely an envelope of holomorphy* (*or meromorphy*) *of* (D, ϕ) *with respect to* $\mathcal{F}$.

A domain (D, ϕ) over a complex manifold M is called *pseudoconvex*, if for every point $p \in M$ there exists a neighborhood U of p such that $\phi^{-1}(U)$ is a Stein manifold.

Proposition 2 (Kajiwara [2]). *Let* (D, ϕ) *be a domain over* M *and* $(\tilde{\lambda}, \tilde{D}, \tilde{\phi})$ *be the envelope of holomorphy* (*or meromorphy*) *of* (D, ϕ) *with respect to* $\mathcal{F}$. *Then* $(\tilde{D}, \tilde{\phi})$ *is pseudoconvex*.

§ 2. Domain over Grassmann manifold. The *Grassmann manifold* G_k is defined to be the set of k-dimensional linear subspace of $\mathbf{C}^n$ for $k < n$. Recently, Ueda [10] obtained the following

Proposition 3. *Let* (D, ϕ) *be a pseudoconvex domain over a Grassmann manifold* G. *If* ϕ *is not a biholomorphic mapping of* D *onto* G, *then* D *is a Stein manifold*.

Using this result, we obtain the following

Proposition 4. *Let* (D, ϕ) *be a domain over a Grassmann manifold* G *and* $(\tilde{\lambda}, \tilde{D}, \tilde{\phi})$ *be the envelope of holomorphy* (*or meromorphy*) *of* (D, ϕ)

with respect to F. *If* $\tilde{\phi}$ *is not a biholomorphic mapping of* $\tilde{D}$ *onto* G, *then* $\tilde{D}$ *is a Stein manifold.*

Proof. By Proposition 2, $(\tilde{D}, \tilde{\phi})$ is a pseudoconvex domain over G. Since $\tilde{\phi}$ is not a biholomorphic mapping of $\tilde{D}$ onto G, $\tilde{D}$ is a Stein manifold by Proposition 3. Q.E.D.

Proposition 5. *Let* (D, ϕ) *be a domain over a Grassmann manifold* G *and* f *be a meromorphic function on* D. *If* f *can not be meromorphically continued to* G, *then there exist holomorphic functions* g *and* h *on* D *such that* $f = g/h$ *on* D.

Proof. Let $(\tilde{\lambda}, \tilde{D}, \tilde{\phi})$ be the domain of meromorphy of f and $\tilde{f}$ be the meromorphic continuation of f to $(\tilde{\lambda}, \tilde{D}, \tilde{\phi})$. Since f can not be meromorphically continued to G, $\tilde{\phi}$ is not a biholomorphic mapping of $\tilde{D}$ onto G. Therefore $\tilde{D}$ is a Stein manifold by Proposition 4. By Hitotumatu-Kota [1] there exist holomorphic functions $\tilde{g}$ and $\tilde{h}$ on $\tilde{D}$ such that $\tilde{f} = \tilde{g}/\tilde{h}$ on $\tilde{D}$. If we put $g = \tilde{g}\circ\tilde{\lambda}$ and $h = \tilde{h}\circ\tilde{\lambda}$, g and h are holomorphic functions on D such that $f = g/h$ on D. Q.E.D.

§ 3. Cousin-II domains. Let D be a domain in a complex manifold M and $U = \{U_j\}$ be an open covering of D. $\Gamma = \{(U_j, f_j)\}$ is called a *Cousin*-II *distribution on* D if every f_j is a meromorphic function on U_j and every f_j/f_k is a non-vanishing holomorphic function on $U_j \cap U_k$ $(\neq \emptyset)$. A meromorphic function f on D is called a *solution of* Γ if every f/f_j is a non-vanishing holomorphic function on U_j. D is called a *Cousin*-II *domain* if every Cousin-II distribution on D has a solution.

Proposition 6 (Siegel [9]). *Let* D *be a Cousin*-II *domain. Then for* *every meromorphic function* f *on* D *there exist holomorphic functions* g *and* h *on* D *such that* $f = g/h$ *on* D *and that* g *and* h *are relatively prime at every point of* D.

The following Lemma is used in our proof of the theorem.

Lemma. *Let* D *be a Cousin*-II *domain in a Grassmann manifold* G. *Then for every point* $p \neq q$ *in* D *there exists a holomorphic function* g *on* D *such that* $g(p) \neq g(q)$.

Proof. Since Grassmann manifold G is projective algebraic, G is embedded by a holomorphic mapping τ into an m-dimensional projective space P^m. Let $(z_0, z_1, \cdots, z_m)$ be the homogeneous coordinates of P^m. If we put $\tau(p) = (\xi_0, \xi_1, \cdots, \xi_m)$ and $\tau(q) = (\eta_0, \eta_1, \cdots, \eta_m)$, without loss of generality we may suppose that $\xi_0 \neq 0$, $\xi_1 = 0$ and $\eta_0 = 0$, $\eta_1 \neq 0$ because of $\tau(p) \neq \tau(q)$. We consider the meromorphic function

$$f = \frac{z_0}{z_1} \circ \tau$$

on D. Then we have $1/f(p) = 0$ and $f(q) = 0$. Since D is a Cousin-II domain, by Proposition 6 there exist holomorphic functions g and h such that $f = g/h$ on D and that g and h are relatively prime at every point of D. Since $f(q) = g(q)/h(q) = 0$, we have $g(q) = 0$. On the other hand, $1/f$ is a holomorphic function on a neighborhood of p and $1/f(p) = h(p)/g(p) = 0$. Since g and h are relatively prime at p, we have $g(p) \neq 0$. Therefore we have $g(p) \neq g(q)$. Q.E.D.

§ 4. Domain with vanishing cohomology groups. We now show the following main result.

Theorem. *Let D be a domain in an n-dimensional Grassmann manifold G. Then D is a Stein manifold with $H^2(D, Z) = 0$ if it satisfies the conditions $H^1(D, O^*) = 0$ and $H^q(D, O) = 0$ for $2 \le q \le n-1$.*

Proof. Let M^* be the sheaf of germs of invertible meromorphic functions. From the exact sequence

$$0 \to O^* \to M^* \to M^*/O^* \to 0$$

we obtain the exact cohomology sequence

(1) $$\cdots \to H^0(D, M^*) \to H^0(D, M^*/O^*) \to H^1(D, O^*) \to \cdots.$$

Since $H^1(D, O^*) = 0$ by our assumption, we have that D is a Cousin-II domain by the exact sequence (1).

Assume that D is not a Stein manofold. Then by Proposition 1 there exists uniquely the envelope of holomorphy $(\tilde{D}, \tilde{\phi})$ of D. Using Lemma, we have a non-constant holomorphic function g on D. It then follows that $\tilde{\phi}$ is not a biholomorphic mapping of $\tilde{D}$ onto G and $\tilde{D}$ is a Stein manifold by Proposition 4. Since $H^0(D, O)$ separates points by Lemma, D may be considered as an open subset of $\tilde{D}$. The domain D in the Stein manifold $\tilde{D}$ satisfies $H^1(D, O^*) = 0$ and $H^q(D, O) = 0$ for $2 \le q \le n-1$ by our assumption. So D is a Stein manofold by Mori [7], but this contradicts our assumption. Therefore D must be a Stein manifold.

From the exact sequence

$$0 \to Z \to O \to O^* \to 0$$

we obtain the exact cohomology sequence

(2) $$\cdots \to H^1(D, O) \to H^1(D, O^*) \to H^2(D, Z) \to H^2(D, O) \to \cdots.$$

By our assumption $H^1(D, O^*) = H^2(D, O) = 0$ we have $H^2(D, Z) = 0$. Q.E.D.

Corollary 1. *Let D be a domain in an n-dimensional Grassmann manifold*

G. Then D is a Stein manifold if it satisfies $H^q(D, \mathcal{O}) = 0$ for $1 \le q \le n-1$ and $H^2(D, Z) = 0$.

Proof. By the above exact sequence (2) we have $H^1(D, \mathcal{O}^*) = 0$. Therefore D is a Stein manifold by Theorem. Q.E.D.

Corollary 2. *Let (D, ϕ) be a domain over an n-dimensional Grassmann manifold G such that $H^0(D, \mathcal{O})$ separates points of D. Then D is a Stein manifold if it satisfies $H^1(D, \mathrm{A}_L) = 0$ for a complex Lie group L and $H^q(D, \mathcal{O}) = 0$ for $2 \le q \le n-1$.*

Proof. As in the proof of Theorem, we may show that D is a Stein manifold. Q.E.D.

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