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1. Introduction

Let P and Q be polyhedra. For a map f from P to Q , we denote the mapping cylinder of f by C_f , that is,

$$C_f = ((P \times I) + Q) / (f(p) = p \times 1, p \in P),$$

where I means the closed unit interval $[0, 1]$ and $+$ means the disjoint union. The standard simplicial structure of C_f is explained later.

Let W be a manifold and P a subpolyhedron in the interior $\overset{\circ}{W}$ of W . By $N(P, W)$, we denote the regular neighborhood of P in W . It is well-known that there exists a map f from $\dot{N}(P, W)$ to P satisfying $(N(P, W), P) = (C_f, P)$, where $\dot{N}(P, W)$ means the boundary of $N(P, W)$ (see for example [1]).

In this paper, we are interested in the following problem.

Problem. Let W and P be a manifold and a polyhedron, respectively, and let f be a map from W to P . Find a necessary and sufficient condition on f for C_f to be a manifold.

In [1], M. Cohen obtained the following results.

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Theorem 7.1 [1]. Assume that W and P are a manifold and a polyhedron, respectively, and that f is a map from W to P . If f is dual collapsible, there exists a homeomorphism from the pair $(W \times I, W \times 0)$ to (C_f, W) which is the identity on $W \times 0$.

Theorem 8.1 [1]. Under the same conditions on W, P and f , if C_f is a manifold and P is contained in C_f , then f is dual collapsible.

The results of this paper are stated as follows:

Theorem 1. Let W be a closed n -manifold and P a polyhedron with $\dim P < n$. Suppose that f is a map from W to P such that C_f is a manifold. Then, for any vertex v of P , there exist a map g from $N(f^{-1}(v), W)$ to $\text{lk}(v, P)$ and an embedding h from C_g to S^n satisfying

- (1) C_g is a manifold,
- (2) $N(f^{-1}(v), W) = S^n - (h(C_g))^\circ$,
- (3) $(S^n, \text{lk}(v, P)) = (\text{lk}(v, C_f), \text{lk}(v, P))$.

Corollary. If further, P is a p -manifold, $H_*(f^{-1}(v)) = H_*(S^{n-p-1})$ for any vertex v of P .

Theorem 2. Let W be a closed n -manifold with $n \geq 2$ and P a 1-polyhedron and let f be a map from W to P . Then, C_f is a manifold if and only if $N(f^{-1}(v)) = S^n - (\#vB^n)^\circ$ for any vertex v of P , where $\#v$ means the number of the connected components of $\text{lk}(v, P)$ and $\#vB^n$ means the disjoint $\#v$ n -balls in S^n .

2. Preliminaries.

First of all, we have to review some of the results in [1]. For the unexplained definitions and notations, see [1].

Definition 1. Let f be a simplicial map between simplicial complexes K and L . Suppose that K and L are embedded in some euclidean space and are joinable there. Let L' be the standard barycentric subdivision of L and K' a first derived subdivision of K chosen so that f is also simplicial as a map from K' to L' . The standard simplicial structure of C_f is defined as follows.

$$C_f = \{\alpha b(\sigma) \cdots b(\tau) \mid \alpha < f(\sigma), \sigma < \cdots < \tau \in K\} + L,$$

where $b(\sigma)$ denotes the barycenter of σ used to make K' from K , $\alpha < f(\sigma)$ means that α is a face of $f(\sigma)$ and $\alpha b(\sigma) \cdots b(\tau)$ means the simplex obtained by joining α and $b(\sigma) \cdots b(\tau)$ which is the simplex of K' with vertices $b(\sigma), \dots, b(\tau)$.

For a simplicial map f , the mapping cylinder with this structure is called the simplicial mapping cylinder of f and is denoted by the same symbol C_f .

Remark. The simplicial mapping cylinder C_f is a subcomplex of $K'L$ and can be written as follows.

$$C_f = \{f(\psi)b(\sigma) \cdots b(\tau) \mid \psi < \sigma < \cdots < \tau \in K\} + L.$$

Definition 2. Suppose that $f: K \rightarrow L$ and $f: K' \rightarrow L'$ be simplicial maps as in Definition 1. For a simplex α of L , we define the dual to α with respect to f , written $D(\alpha, f)$, by

$$D(\alpha, f) = \{b(\sigma) \cdots b(\tau) \mid \alpha < f(\sigma), \sigma < \cdots < \tau \in K\}.$$

Put

$$D(\alpha, f) = \{b(\sigma) \cdots b(\tau) \mid \alpha \not< f(\sigma), \sigma < \cdots < \tau \in K\}.$$

Remark. For a simplicial complex L and a simplex α of L , $D(\alpha, L)$ denotes the usual dual to α in L .

We may state the following propositions without proof. For the proofs of them, see [1].

Proposition 1. $D(\alpha, f) = \text{lk}(\alpha, C_f) \cap K'$.

Proposition 2. $D(\alpha, f) = f^{-1}(D(\alpha, L))$.

Proposition 3. Suppose that K is a closed (combinatorial) n -manifold and α is an i -simplex of $f(K)$. Then, $D(\alpha, f)$ is an $(n - i)$ -manifold with boundary $D(\alpha, f) = d(\alpha, f)$.

Proposition 4. $D(\alpha, f) \times f^{-1}(b(\alpha))$, that is, $D(\alpha, f)$ collapses to $f^{-1}(b(\alpha))$.

Proposition 5. For any simplicial map $f: K \rightarrow L$ between simplicial complex K and L , $C_f \times L$.

Proposition 6. If f is a simplicial map from an n -simplex onto a simplex (of any dimension), $C_f = B^{n+1}$, an $(n + 1)$ -ball.

3. A necessary condition.

For a PL-map $f: P \rightarrow Q$ between polyhedra P and Q , we define the PL-mapping cylinder of f , denoted by the same symbol C_f , as follows: Let K and L be simplicial structures of P and Q , respectively, such that $f: K \rightarrow L$ is simplicial. Then, the PL-mapping cylinder C_f is defined to be the underlying polyhedron of the simplicial mapping cylinder of $f: K \rightarrow L$.

It is not hard to see that the definition of the PL-mapping cylinder is

independent from the choice of K and L .

Hereafter, maps and manifolds mean PL -maps and PL -manifolds, and mapping cylinders are also PL -ones, unless otherwise stated.

Lemma 1. *Let P and Q be polyhedra and $f: P \rightarrow Q$ a map such that C_f is a manifold. If $\dim P \geq \dim Q$, f is onto.*

Proof. Let K and L be simplicial structures of P and Q such that $f: K \rightarrow L$ is simplicial. Suppose that f is not onto. Then, there exists a vertex v in $L - f(K)$. Taking a barycentric subdivision L' of L , we have $\text{st}(v, L') \subset L - f(K)$. This implies $\dim \text{st}(v, L) \leq \dim Q \leq \dim P = \dim C_f - 1$. This contradicts our hypothesis that C_f is a manifold.

Lemma 2. *Let W be a closed n -manifold and P a polyhedron with $\dim P < n$. Suppose that $f: W \rightarrow P$ is a map such that C_f is a manifold. Then, $\dot{C}_f = W$, and hence $P \subset \dot{C}_f$.*

Proof. By [1], we have $\dot{C}_f \subset W \cup P$, because W is closed. If $\dot{C}_f \cap P \neq \emptyset$, then P should contain an n -manifold, because $W \cap P = \emptyset$. This is impossible, for $\dim P < n$. Thus, $\dot{C}_f \subset W$. On the other hand, since W has a neighborhood homeomorphic to $W \times I$ with $W \times 0 = W$ in C_f , $\dot{C}_f \supset W$ follows immediately.

Lemma 3. *Let K and L be simplicial complexes and f an onto map from K to L . Then, for any vertex v of L , $\text{lk}(v, C_f)$ is homeomorphic to $D(v, f) \cup C_g$ keeping $D(v, f)$ fixed, where g is the map from $d(v, f)$ to $f(d(v, f))$, i.e., $g = f|_{d(v, f)}$.*

Proof. Let K' and L' be the subdivisions of K and L chosen as

in Definition 1. Then, we have

$$\text{st}(v, C_f) = \{\alpha b(\sigma) \cdots b(\tau) \mid v < \alpha < f(\sigma), \sigma < \cdots < \tau \in K\} \cup D(v, f),$$

and

$$\text{lk}(v, C_f) = \{\alpha b(\sigma) \cdots b(\tau) \mid v\alpha < f(\sigma), \sigma < \cdots < \tau \in K\} \cup D(v, f),$$

for a vertex v of L . We construct a subdivision of $\text{st}(v, C_f)$. Let $\alpha^p = b(\sigma_0) \cdots b(\sigma_p)$ be a p -simplex of $D(v, f)$ satisfying $\sigma_0 < \cdots < \sigma_p$. Define $\gamma(\alpha^0) = \{b(\sigma_0)f(\alpha^0) \text{ and the faces}\}$, where $\alpha^0 = b(\sigma_0)$. By induction on p , we define $\gamma(\alpha^i) = \{b(\sigma_i)(f(\alpha^i) \cup \gamma(\alpha^{i-1})) \text{ and the faces}\}$, where α^{i-1} means the opposite face of $b(\sigma_i)$ in the simplex $\alpha^i = b(\sigma_0) \cdots b(\sigma_i)$. Then,

$$K' \cup L' \cup \bigcup_{v \in L} \bigcup_{\alpha^p \in D(v, f)} \gamma(\alpha^p)$$

gives a subdivision C'_f of C_f . C'_f induces a triangulation T of the topological mapping cylinder of g . It is not hard to see that T is PL -homeomorphic to C_g . Thus, we use the same notation C_g instead of T . Then, we have $\text{lk}(v, C'_f) = D(v, f) \cup C_g$. By the pseudo radial projection from v , we immediately obtain a required homeomorphism, completing the proof.

Lemma 4. *Let W be a closed n -manifold with $n \geq 1$ and let v be a point. Then, for a map $f: W \rightarrow v$, C_f is an $(n+1)$ -ball if and only if $W = S^n$, an n -sphere.*

Proof. If $W = S^n$, it follows immediately that C_f is an $(n+1)$ -ball, because $C_f = vW$. Suppose that C_f is an $(n+1)$ -ball. We have $\dot{C}_f = W$, by Lemma 2. Therefore, W should be an n -sphere S^n .

Theorem 1. *Let W be a closed n -manifold and P a polyhedron with*

$\dim P < n$. Suppose that $f: W \rightarrow P$ is a map such that C_f is a manifold. Then, for any vertex v of P , there exist a map $g: N(f^{-1}(v), W) \rightarrow \text{lk}(v, P)$ and an embedding $h: C_g \rightarrow S^n$ satisfying the following three conditions.

- (1) C_g is a manifold,
- (2) $N(f^{-1}(v), W) = S^n - (h(C_g))^\circ$,
- (3) $(S^n, h(\text{lk}(v, P))) = (\text{lk}(v, C_f), \text{lk}(v, P))$.

Proof. Since W is a closed n -manifold, $\dim P < n$ and C_f is a manifold, we have $P \subset \overset{\circ}{C}_f$, by Lemma 2. Then, for any vertex v of P , $\text{lk}(v, C_f)$ is an n -sphere S^n , because C_f is an $(n+1)$ -manifold. Taking suitable simplicial structures of W and P , we have $\text{lk}(v, C_f) = D(v, f) \cup C_g$, by Lemma 3. By Proposition 3 and Proposition 4, $D(v, f)$ can be regarded as a regular neighborhood of $f^{-1}(v)$ in W . Then, C_g should be an n -manifold, because $D(v, f) \cap C_g = \dot{D}(v, f)$. Now, we choose h to be the inclusion map of C_g in $\text{lk}(v, C_f) = S^n$. Then, $N(f^{-1}(v), W)$ is homeomorphic to $D(v, f) = S^n - (h(C_g))^\circ$. We can prove (3) by Lemma 3, easily, completing the proof of Theorem 1.

Corollary 1. *If, further, P is a p -manifold, then*

$$H_*(f^{-1}(v)) = H_*(S^{n-p-1})$$

for any vertex v of P , provided that P is closed.

Proof. Since $H_*(f^{-1}(v)) = H_*(N(f^{-1}(v), W)) = H_*(S^n - (h(C_g))^\circ)$ and $h(C_g)$ can be regarded as a regular neighborhood of $\text{lk}(v, P)$ in S^n , we obtain the conclusion by the Alexander duality.

The following corollary is the necessary part of Theorem 2 in the intro-

duction.

Corollary 2. *Let W be a closed n -manifold with $n \geq 2$, and let P be a 1-polyhedron. Suppose that $f: W \rightarrow P$ is a map such that C_f is a manifold. Then, $N(f^{-1}(v), W) = S^n - (\#vB^n)^\circ$ for any vertex v of P .*

Proof. By Theorem 1, we have $\text{lk}(v, C_f) = S^n - (h(C_g))^\circ$, for any vertex v of P . Since $\text{lk}(v, P)$ consists of $\#v$ points and C_g is a manifold, it follows that C_g is the disjoint union of $\#v$ n -balls by Proposition 5. Thus, $h(C_g) = \#vB^n$.

4. A sufficient condition.

We prove the sufficiency of Theorem 2.

Lemma 5. *Let W be a closed n -manifold with $n \geq 1$, and let P be a 1-polyhedron. Suppose that $f: W \rightarrow P$ is a map satisfying the condition $N(f^{-1}(v), W) = S^n - (\#vB^n)^\circ$ for any vertex v of P . Then, C_f is a manifold.*

Proof. It follows immediately that $\text{lk}(v, C_f)$ is an n -ball for any vertex v of W , because W is a manifold. Thus, it is sufficient to show that $\text{lk}(v, C_f)$ is an n -sphere for any vertex v of P for C_f to be a manifold. We have $\text{lk}(v, C_f) = D(v, f) \cup C_g$ by Lemma 3 and $D(v, f) = N(f^{-1}(v), W)$ for any vertex v of P . It suffices to prove $C_g = \#vB^n$, because $N(f^{-1}(v), W) = S^n - (\#vB^n)^\circ$ by the assumption. Note that $\text{lk}(v, P)$ consists of $\#v$ points and $D(v, f)$ is the disjoint union of $\#v$ $(n-1)$ -spheres. Since g is onto, we have $C_g = \#vB^n$, by Lemma 4.

Thus, we proved Theorem 2.

5. Other dimensional cases.

We can prove the following Theorem 3 by the similar argument to that used to prove Theorem 1.

Theorem 3. *Let W be a closed n -manifold and P an n -polyhedron. Suppose that $f: W \rightarrow P$ is a map such that C_f is a manifold containing P in $\overset{\circ}{C}_f$. Then, the same conclusion as Theorem 1 holds.*

Corollary 3. *If, further, P is a closed manifold, then $f^{-1}(v)$ is the disjoint union of exactly two contractible polyhedra for any vertex v of P .*

Theorem 4. *Let W be a closed 2-manifold and P a homogeneous 2-polyhedron without free face. Then, for a map $f: W \rightarrow P$, C_f is a manifold containing P in C_f if and only if $N(f^{-1}(v), W)$ consists of rank $H_1(\text{lk}(v, P)) + 1$ disjoint 2-balls and C_g is a 2-manifold for any vertex v of P .*

Proof. The proof is similar to those of Corollary 2 and Lemma 5 by making use of Theorem 3. It remains to show that C_g can be embedded in S^2 , in the proof of sufficiency. This can be proved from the facts that C_g is an abstract regular neighborhood of $\text{lk}(v, P)$ and the boundary $\overset{\circ}{C}_g$ consists of rank $H_1(\text{lk}(v, P)) + 1$ disjoint circles.

Reference

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