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On Some Recent Results on Fixed Points III

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In this note we shall generalize fixed point theorems obtained recently by Browder [1] and Fisher [2]. Incidentally, it will be shown that a recent result of Fisher-Khan [3] can be derived from a slight improvement of a result in [4]. Throughout this note, ω and $\mathbb{R}_+$ denote the set of all non-negative integers and that of all nonnegative real numbers, respectively.

1. In his paper [1], Browder has proved several fixed point theorems with the help of a *contractive gauge function* which is, by definition, a nonnegative real valued function Φ on $\mathbb{R}_+$ satisfying the following conditions:¹⁾

- (1) $\Phi(t) < t$ for all $t > 0$.
- (2) $\Phi(0) = 0$.
- (3) Φ is nondecreasing.
- (4) Φ is continuous from the right.

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¹⁾ Obviously (1) and (3) imply (2). Further (1) and (4) also imply (2). In fact, if $\alpha = \Phi(0) > 0$, there is a $\delta > 0$ such that $\alpha/2 < \Phi(t) < 3\alpha/2$ whenever $0 \leq t \leq \delta$, and so putting $t = \min\{\delta, \alpha/2\}$, we have $\Phi(t) < t \leq \alpha/2 < \Phi(t)$, a contradiction.

However all the fixed point theorems in [1] remain valid for a nonnegative real valued function Φ on $\mathbb{R}_+$ which need not satisfy (3). We begin with the following theorem which extends Proposition 2 of [1].

Theorem 1. *Let f be a continuous mapping of a complete metric space (X, d) into itself which carries a subset M of X into M . Let Φ be a nonnegative real valued function on $\mathbb{R}_+$ which is upper semicontinuous from the right on $\mathbb{R}_+ \setminus \{0\}$ and satisfies the condition (1). If each point of M is regular²⁾ for f and if there exists a positive integer valued function n on M such that*

$$(5) \quad D(f^n(x)) \leq \Phi(D(x)),$$

for all $x \in M$, then for each $x_0 \in M$, the sequence $\{f^n(x_0)\}_{n \in \omega}$ converges to a fixed point of f , where $D(x)$ denotes the diameter of the orbit $O_f(x) = \{f^n(x) \mid n \in \omega\}$ of x under f .

Proof. The recursion theorem guarantees the existence of a sequence $\{x_m\}_{m \in \omega}$ such that $x_{m+1} = f^{n(x_m)}(x_m)$ for all $m \in \omega$. We shall show that the sequence $s = \{D(x_m)\}_{m \in \omega}$ converges to 0. Since s is nonincreasing, it converges to some $\alpha \in \mathbb{R}_+$. Suppose $\alpha > 0$. Then $\Phi(\alpha) < \alpha$, and so there exists a $\rho > 0$ such that $\Phi(t) < \alpha$ whenever $\alpha \leq t < \alpha + \rho$. But then we can find an $m \in \omega$ such that $\alpha \leq D(x_m) < \alpha + \rho$. Consequently (5) implies

$$\alpha \leq D(x_{m+1}) \leq \Phi(D(x_m)) < \alpha,$$

a contradiction. Therefore $\alpha = 0$, and thus s converges to 0, as desired. Since each x_m belongs to the orbit of x_0 under f , this means that $t = \{f^n(x_0)\}_{n \in \omega}$ is a Cauchy sequence. Hence t converges to some $a \in X$. So

²⁾ A point x of a metric space (X, d) is said to be *regular* for a selfmapping f on X if the orbit of x under f is bounded (see [5]).

the continuity of f implies that $\{f^{n+1}(x_0)\}_{n \in \omega}$ converges to $f(a)$. It follows that $f(a) = a$.

The following corollary improves Proposition 3 of [1].

Corollary 1. *Let f be a continuous mapping of a complete metric space (X, d) into itself which carries a nonempty subset M of X into M . Let Φ be a nonnegative real valued function on $\mathbb{R}_+$ which is upper semicontinuous from the right on $\mathbb{R}_+ \setminus \{0\}$ and satisfies the condition (1). If each point of M is regular for f and if there exists a positive integer valued function n on M such that*

$$\delta(O_f(f^{n(x)+n(y)}(x)) \cup O_f(f^{n(x)+n(y)}(y))) \leq \Phi(\delta(O_f(x) \cup O_f(y)))$$

for all $x, y \in M$, then f has a unique fixed point, and for any $x \in M$, the sequence $\{f^n(x)\}_{n \in \omega}$ converges to the fixed point, where $\delta(A)$ denotes the diameter of $A \subset X$.

Proof. For every $x \in M$, we have

$$D(f^{2n(x)}(x)) = \delta(O_f(f^{n(x)+n(x)}(x))) \leq \Phi(\delta(O_f(x))) = \Phi(D(x)),$$

and so Theorem 1 shows that, for any $x \in M$, the sequence $\{f^n(x)\}_{n \in \omega}$ converges to a fixed point of f . Since M is nonempty, this guarantees the existence of a fixed point of f . If $x, y \in X$ are fixed points of f then

$$\begin{aligned} d(x, y) &= \delta(O_f(f^{n(x)+n(y)}(x)) \cup O_f(f^{n(x)+n(y)}(y))) \\ &\leq \Phi(\delta(O_f(x) \cup O_f(y))) \\ &= \Phi(d(x, y)) \end{aligned}$$

and hence $d(x, y) = 0$ by (1). Therefore f has a unique fixed point $a \in X$. Thus, for any $x \in M$, the sequence $\{f^n(x)\}_{n \in \omega}$ must converges to a .

As a generalization of Theorem 2 of [1], we have the following:

Theorem 2. Let f be a continuous mapping of a complete metric space (X, d) into itself which carries a nonempty subset M of X into M . Let Φ be an upper semicontinuous nonnegative real valued function of $\mathbb{R}_+$ satisfying the conditions (1) and (2). Suppose that each point of M is regular for f and that there exists a positive integer valued function n on M satisfying the following condition: For each $x \in M$, if $m \geq n(x)$ then for every $y \in M$, there exist subsets J_1, J_2 and J_3 of $\omega \times \omega$ such that $J_1 \cup J_2 \cup J_3 \neq \emptyset$ and

$$(6) \quad d(f^m(x), f^m(y)) \leq \Phi(\max\{\sigma_1, \sigma_2, \sigma_3\})$$

where $\sigma_1 = \sup\{d(f^p(x), f^q(y)) \mid (p, q) \in J_1\}$, $\sigma_2 = \sup\{d(f^p(x), f^q(x)) \mid (p, q) \in J_2\}$ and $\sigma_3 = \sup\{d(f^p(y), f^q(y)) \mid (p, q) \in J_3\}$. Then f has a unique fixed point, and for any $x \in M$ the sequence $\{f^n(x)\}_{n \in \omega}$ converges to the fixed point.

Proof. Let $x_0 \in M$. Then by the recursion theorem, there exists a sequence $\{x_m\}_{m \in \omega}$ in M such that $x_{m+1} = f^{n(x_m)}(x_m)$ for all $m \in \omega$. Since the sequence $\{D(x_m)\}_{m \in \omega}$ is nonincreasing, it converges to some α in $\mathbb{R}_+$. Suppose $\alpha > 0$. Then $\Phi(\alpha) < \alpha$, and hence $\Phi(\alpha) < \alpha - \varepsilon$ for some $\varepsilon > 0$. Since Φ is upper semicontinuous, we can find a ρ with $0 < \rho < \varepsilon$ such that $\Phi(t) < \alpha - \varepsilon$ whenever $\alpha - \rho < t < \alpha + \rho$. Choose an $m \in \omega$ satisfying $D(x_m) < \alpha + \rho$. Let u and v be arbitrary points of $O_{f^{n(x_m)}}(x_m)$, the orbit of x_{m+1} under f . Then $u = f^{n(x_m)+i}(x_m)$ and $v = f^{n(x_m)+j}(x_m)$ for some $i, j \in \omega$, which may be assumed to be $i \geq j$. There exist, for $k = n(x_m) + j$ and $y = f^{i-j}(x_m)$, subsets J_1, J_2 and J_3 of $\omega \times \omega$ such that $J_1 \cup J_2 \cup J_3 \neq \emptyset$ and

$$d(u, v) = d(f^k(x_m), f^k(y)) \leq \Phi(\max\{\sigma_1, \sigma_2, \sigma_3\})$$

where $\sigma_1 = \sup\{d(f^p(x_m), f^q(y)) \mid (p, q) \in J_1\}$, $\sigma_2 = \sup\{d(f^p(x_m), f^q(x_m)) \mid$

$(p, q) \in J_2\}$, and $\sigma_3 = \sup\{d(f^p(y), f^q(y)) \mid (p, q) \in J_3\}$. Since y belongs to $O_f(x_m)$, we have $\sigma = \max\{\sigma_1, \sigma_2, \sigma_3\} \leq D(x_m)$. If $\alpha - \rho \leq \sigma$ then $\alpha - \rho \leq \sigma \leq D(x_m) < \alpha + \rho$ and hence $d(u, v) \leq \Phi(\sigma) < \alpha - \varepsilon < \alpha - \rho$. If $0 < \sigma \leq \alpha - \rho$ then $d(u, v) \leq \Phi(\sigma) < \sigma \leq \alpha - \rho$. If $\sigma = 0$ then $d(u, v) \leq \Phi(0) = 0 < \alpha - \rho$. Therefore we have $d(u, v) < \alpha - \rho$. Since $u, v \in O_f(x_{m+1})$ are arbitrary, this yields $D(x_{m+1}) \leq \alpha - \rho < \alpha$ which is impossible. Consequently $\alpha = 0$, and thus $D(x_m) \rightarrow 0$ as $m \rightarrow \infty$. Since each x_m is in the orbit of x_0 under f , this means that $\{f^n(x_0)\}_{n \in \omega}$ is a Cauchy sequence. Hence it converges to some point of X which is a fixed point of f , because of the continuity of f . Since M is nonempty, this establishes the existence of a fixed point of f . It will suffice to prove the uniqueness of fixed point of f . Let $x, y \in X$ be fixed points of f . Then there exist subsets J_1, J_2 and J_3 of $\omega \times \omega$ such that $J_1 \cup J_2 \cup J_3 \neq \emptyset$ and

$$d(x, y) = d(f^{n(x)}(x), f^{n(y)}(y)) \leq \Phi(\max\{\sigma_1, \sigma_2, \sigma_3\})$$

where $\sigma_1 = \sup\{d(f^p(x), f^q(y)) \mid (p, q) \in J_1\}$, $\sigma_2 = \sup\{d(f^p(x), f^q(x)) \mid (p, q) \in J_2\}$ and $\sigma_3 = \sup\{d(f^p(y), f^q(y)) \mid (p, q) \in J_3\}$. If $\sigma = \max\{\sigma_1, \sigma_2, \sigma_3\} > 0$ then we have $d(x, y) \leq \Phi(\sigma) < \sigma \leq d(x, y)$, a contradiction. Therefore $\sigma = 0$, and thus we have $d(x, y) \leq \Phi(0) = 0$ or $x = y$.

The following corollary corresponds to Theorem 1 of [1]:

Corollary 2. *Let f be a continuous mapping of a complete metric space (X, d) into itself which carries a nonempty subset M of X into M . Let Φ be an upper semicontinuous nonnegative real valued function on $\mathbb{R}_+$ satisfying the conditions (1) and (2). Suppose that each point of M is regular for f and that there exist a positive integer m and subsets J_1, J_2 and J_3 of $\omega \times \omega$ such that (6) holds for all $x, y \in X$. Then f has*

a unique fixed point, and for any $x \in M$ the sequence $\{f^n(x)\}_{n \in \omega}$ converges to the fixed point.

2. The following theorem is obtained by Fisher [2]:

Theorem. Let S and T be continuous mappings of a nonempty compact metric space (X, d) into itself satisfying the following conditions for all x, y in X :

$$(7) \quad d(STx, TSy) < \max\{d(x, Tx), d(y, Sy)\} \quad \text{if} \quad \max\{d(x, Tx), d(y, Sy)\} \neq 0.$$

$$(8) \quad d(STx, TSy) = 0 \quad \text{if} \quad \max\{d(x, Tx), d(y, Sy)\} = 0.$$

Then each of S and T has a fixed point. Further, if S and T have a common fixed point, then it is unique.

However the conclusion of this theorem can be strengthened. In fact we have the following theorem:

Theorem 3. Let S and T be continuous mappings of a nonempty compact metric space (X, d) into itself. If the condition (7) is satisfied for all x, y in X , then S and T have a common fixed point; further if (8) is satisfied, it is unique.

Proof. Let f denote the nonnegative real valued function

$$x \mapsto \max\{d(x, Tx), d(Tx, STx)\}$$

defined on X . Since f is continuous, it assumes the least value at some $a \in X$. If $f(a) = 0$ then $Ta = a$ and $STa = Ta$, and thus a is a common fixed point of S and T . Suppose $f(a) > 0$. Then it follows from (7) that

$$(*) \quad d(STa, TSTa) < f(a).$$

We shall show

$$\alpha = \max \{d(STa, TSTa), d(Ta, STa)\} = 0,$$

which implies that Ta is a common fixed point of S and T . To this end, suppose $\alpha > 0$. Then by (7) we have

$$d(STSTa, TSTa) < \alpha.$$

Hence if $\alpha = d(STa, TSTa)$, we obtain a contradiction,

$$\begin{aligned} f(STa) &= \max \{d(STa, TSTa), d(TSTa, STSTa)\} \\ &= d(STa, TSTa) \\ &< f(a), \end{aligned}$$

by (*). Therefore $\alpha \neq d(STa, TSTa)$. But then we have

$$\begin{aligned} f(STa) &= \max \{d(STa, TSTa), d(TSTa, STSTa)\} \\ &< \alpha = d(Ta, STa) \\ &\leq f(a) \end{aligned}$$

which is absurd. Thus $\alpha = 0$. If (8) is satisfied and if $x, y \in X$ are common fixed points of S and T , then $d(x, y) = d(STx, TSy) = 0$. This completes the proof.

3. The following theorem improves Theorem 1 of [4]:

Theorem 4. *Let f be a mapping of an L-space $(X, \rightarrow)$ into itself, and d a lower semicontinuous nonnegative real valued function on the L-space $(X \times X, \Rightarrow)$, where $\{(x_n, y_n)\}_{n \in \omega} \Rightarrow (x, y)$ means $\{x_n\}_{n \in \omega} \rightarrow x$ and $\{y_n\}_{n \in \omega} \rightarrow y$. If there exist a compact subset K of X and a selfmapping g on K such that f is continuous on K and*

$$d(fg(x), g(x)) < d(f(x), x)$$

for every $x \in K$ with $f(x) \neq x$, then f has a fixed point.

Proof. Since the mapping $x \mapsto d(f(x), x)$ is lower semicontinuous on K by Lemma 4 of [4], it assumes the least value at some $a \in K$ by Lemma 2 of

[4]. Therefore $d(f(a), a) \leq d(fg(a), g(a))$, and thus $f(a) = a$.

Using this theorem, we can prove the main result of Fisher-Khan [3]:

Corollary 3. *Let S and T be continuous densifying mappings of a complete metric space (X, d) , and let F_1 and F_2 be nonnegative real valued function on $X \times X$. Suppose that F_1 or F_2 is lower semicontinuous and the following inequalities hold:*

$$F_1(Sx, TSy) < \max \{F_2(x, Sx), F_2(x, Sy), F_2(y, Sy), F_1(Sy, TSy)\}$$

for all $x, y \in X$ with $x \neq Sy$.

$$F_2(Tx, STy) < \max \{F_1(x, Tx), F_1(x, Ty), F_1(y, Ty), F_2(Ty, STy)\}$$

for all $x, y \in X$ with $x \neq Ty$.

If there exists an $x_0 \in X$ such that the sequence $\{x_n\}_{n \in \omega}$ defined by $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ for all $n \in \omega$ is bounded, then S or T has a fixed point.

Proof. We may assume F_1 is lower semicontinuous. Consider the set $A = \{x_{2n+1} \mid n \in \omega\}$. If $\alpha(A)$, the measure of non-compactness of A , is not 0 then we have

$$\alpha(A) = \alpha(\{x_1\} \cup ST(A)) = \alpha(ST(A)) < \alpha(A),$$

a contradiction. Hence $\alpha(A) = 0$, and so $K = \overline{A}$ is compact. Since $ST(A) \subset A$, the continuity of S and T shows that the restriction of ST into K is a mapping of K into K . Let $\rightarrow$ denote the convergence defined by the metric d . Obviously K is a compact subset of the L-space $(X, \rightarrow)$, and T is continuous on K . Suppose now that both S and T have no fixed point. Then the function d^* on $X \times X$ defined by $d^*(x, y) = F_1(y, x)$ is lower semicontinuous on the L-space $(X \times X, \Rightarrow)$, and for all $x \in K$,

$$d^*(TSTx, STx) = F_1(STx, TSTx)$$

$$\begin{aligned}
 &< F_2(Tx, STx) \\
 &< F_1(x, Tx) \\
 &= d^*(Tx, x)
 \end{aligned}$$

in view of the hypothesis. Therefore Theorem 3 shows that T has a fixed point. This contradicts that both S and T have no fixed point. Thus either S or T must have a fixed point.

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