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(Citation)

Mathematics Seminar Notes,8(1):37-55

(Issue Date)

1980

(Resource Type)

journal article

(Version)

Version of Record

(JaLCD0I)

<https://doi.org/10.24546/E0001517>

(URL)

<https://hdl.handle.net/20.500.14094/E0001517>



Representation of Lattice Ordered Autometrized Algebras by Sheaves

By B. V. SUBBA RAO^{*}

Introduction. In [4] we have introduced the concept of a representable autometrized algebra $A = (A, +, \leq, *)$ which includes commutative DR ℓ -semi-groups and hence commutative ℓ -groups, Boolean algebras and Brouwerian algebras as special cases and proved that a lattice ordered autometrized algebra $A = (A, +, \leq, *)$ is a subdirect product of totally ordered representable autometrized algebras if and only if A is representable and

$$\{a * (a \cup b)\} \cap \{b * (a \cup b)\} = 0$$

for all a, b in A . In [5] we have studied the geometry of a special class of representable autometrized algebras $A = (A, +, \leq, *)$ satisfying

$$(R1) \quad a * (a \cap b) + a \cap b = a \quad \text{for all } a, b \text{ in } A, \text{ and}$$

$$(R2) \quad \{a * (a \cup b)\} \cap \{b * (a \cup b)\} = 0 \quad \text{for all } a, b \text{ in } A,$$

which is still wider than the class of commutative DR ℓ -semigroups satisfying " $(a - b) \cap (b - a) \leq 0$ for all a, b in A ". In [6], we have observed that the lattice of all congruence relations on a representable autometrized algebra $A = (A, +, \leq, *)$ satisfying (R1) is a distributive lattice and

Received July 30, 1979.

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that any two congruence relations on A are permutable and also obtained a Chinese remainder theorem for this class of representable autometrized algebras.

As a continuation of our study of representable autometrized algebras, in the present paper, we introduce the notion of a sheaf of representable autometrized algebras and by applying Theorem 2 of U. Maddana Swamy [2], we show that a representable autometrized algebra A satisfying (R1) is isomorphic with the representable autometrized algebra of all global sections with compact supports of a sheaf of homomorphic images of A over a topological space. This completely generalizes the corresponding results of Klaus Keimel [1] for ℓ -rings and of Prabhakara Rao [3] for commutative DR ℓ -semigroups. Keimel [1] has proved that if A is an f -ring, then the space πA of all minimal irreducible ℓ -ideals of A is a zero dimensional Hausdorff space and there exists a T_2 -sheaf $F(A, \pi A)$ of totally ordered rings over πA such that A is isomorphic to an ℓ -subring $\hat{A}$ of sections of $F(A, \pi A)$. Prabhakara Rao [3] has extended this result for the class of representable DR ℓ -semigroups (ie. the class of commutative DR ℓ -semigroups $A = (A, +, \leq, -)$ satisfying $(a - b) \cap (b - a) \leq 0$ for all a, b in A). In the main theorem (see Theorem 43) of this paper, we prove that if $A = (A, +, \leq, *)$ is a representable autometrized algebra satisfying (R1) and (R2), then, the space πA of all minimal prime ideals of A is a zero dimensional Hausdorff space and that there exists a T_2 -sheaf $F(A, \pi A)$ of totally ordered representable autometrized algebras over πA such that A is isomorphic to a subalgebra $\hat{A}$ of the representable autometrized algebra of sections of $F(A, \pi A)$. This theorem generalizes the above mentioned results of Keimel [1] and Prabhakara Rao [3]

and consequently applies to Boolean algebras also. For convenience, we divide this paper into two sections.

§ 1. Throughout this section, $A = (A, +, \leq, *)$ stands for a representable autometrized algebra satisfying the condition

$$(R1) \quad a * (a \cap b) + a \cap b = a \quad \text{for all } a, b \text{ in } A.$$

We freely use the definitions, notations and results from [4], [5] and [6].

Now, we introduce the following

Definition 1. An equivalence relation θ on A is called an *algebraic congruence relation* if and only if θ satisfies

$$(C1) \quad a \equiv b (\theta), c \equiv d (\theta) \text{ imply that } a + c \equiv b + d (\theta) \text{ for any}$$

$$a, b, c, d \text{ in } A,$$

$$(C2) \quad a \equiv b (\theta), c \equiv d (\theta) \text{ imply that } a * c \equiv b * d (\theta) \text{ for any}$$

$$a, b, c, d \text{ in } A,$$

$$(C4) \quad a \equiv b (\theta), c \equiv d (\theta) \text{ imply that } a \cup c \equiv b \cup d (\theta) \text{ for any}$$

$$a, b, c, d \text{ in } A,$$

and

$$(C5) \quad a \equiv b (\theta), c \equiv d (\theta) \text{ imply that } a \cap c \equiv b \cap d (\theta) \text{ for any}$$

$$a, b, c, d \text{ in } A.$$

For convenience, we refer to any usual congruence relation θ on A (as defined in Definition 2.6, [4]) as a metric congruence relation. In [4], we observed that if θ is any metric congruence relation, then θ satisfies (C1), (C2), (C4), (C5) (by Remark 2.6.1, [4]) and (by Definition 2.6, [4])

$$(C3) \quad a \equiv b (\theta) \text{ and } x * y \leq a * b \text{ imply that } x \equiv y (\theta) \text{ for any}$$

$$a, b, x, y \text{ in } A.$$

Now, we have the following

Lemma 2. *Let θ be an equivalence relation on A . Then, θ is a metric congruence relation on A , if and only if it is an algebraic congruence relation on A .*

Proof. If θ is a metric congruence relation, it is clear from above that θ is an algebraic congruence relation. Conversely, assume that θ is an algebraic congruence relation on A and suppose that $a \equiv b (\theta)$ and $x * y \leq a * b$ for some a, b, x, y in A . Since $b \equiv b (\theta)$ and $x * y \equiv x * y (\theta)$, it follows that $a * b \equiv 0 (\theta)$ (by (C2)) and hence $x * y \equiv 0 (\theta)$ (by (C5)). Since $x * (x \cap y) \leq x * y$ and $y * (x \cap y) \leq y * x = x * y$ (by 2.2.2, [4]), it follows that $x * (x \cap y) \equiv 0 (\theta)$ and $y * (x \cap y) \equiv 0 (\theta)$ (by (C5)) and hence $x * (x \cap y) \equiv y * (x \cap y)$. So, since $x \cap y \equiv x \cap y (\theta)$, it follows by (C1) and (R1) that $x \equiv y (\theta)$. This completes the proof.

Lemma 3. *A is a semiregular seminormal autometrized algebra in which the following decomposition property (D) holds.*

(D) $a \leq b + c$ and $a \geq 0, b \geq 0, c \geq 0$ imply that $a = b_1 + c_1$
where $0 \leq b_1 \leq b$ and $0 \leq c_1 \leq c$.

Proof. A is clearly (by Remark 2.4.2, [4]) a semiregular seminormal autometrized algebra. Assume that $a, b, c \in A$ such that $a \geq 0, b \geq 0, c \geq 0$ and $a \leq b + c$. Put $b_1 = a \cap b$ and $c_1 = a * (a \cap b)$. Then, it is clear that $0 \leq b_1 \leq b$. Also, using 2.2.2 and 2.2.1, [4] we have $0 \leq c_1 = a * (a \cap b) = \{a \cap (b + c)\} * \{a \cap b\} \leq (b + c) * b \leq c * 0 = c$. Further by (R1), we have $a = a * (a \cap b) + a \cap b = c_1 + b_1 = b_1 + c_1$. This completes the proof.

Prabhakara Rao (§1.3, [3]) introduced the following concepts of a dual ideal and a prime ideal on an autometrized algebra $B = (B, +, \leq, *)$ and also observed that this notion of prime ideal coincides with the notion of a prime ideal in a Boolean algebra and that of an irreducible ideal introduced by Keimel [1] in an ℓ -ring, when B degenerates into a Boolean algebra and an ℓ -ring respectively.

Definition 4. By a *dual ideal* D of an autometrized algebra $B = (B, +, \leq, *)$ we mean a nonempty subset D of B satisfying the conditions:

(D1) $x \in D$ and $x * 0 \leq m(y * 0)$ for some positive integer m imply that $y \in D$, and

(D2) $x \in D$ and $y \in D$ imply that there exists an element $z \in D$ such that $\langle z \rangle \subset \langle x \rangle \cap \langle y \rangle$ ($\langle b \rangle$ denotes the principal ideal of B generated by $b \in B$).

Definition 5. A proper ideal P of an autometrized algebra $B = (B, +, \leq, *)$ is called a *prime ideal* if and only if $B \setminus P$ is a dual ideal of B .

In view of Lemma 3, all the results (§1.3, [3]) of Prabhakara Rao for semiregular seminormal autometrized algebras with the decomposition property (D) are valid for any representable autometrized algebra $A = (A, +, \leq, *)$ satisfying (R1). However, we state below a few important results among them, some of which are relevant to this paper. In the following, $A = (A, +, \leq, *)$ stands for a representable autometrized algebra satisfying (R1), as mentioned earlier.

Theorem 6. *The following assertions are equivalent for any proper*

ideal P of A .

- (1) P is a prime ideal of A .
- (2) $I \cap J = P$ implies $I = P$ or $J = P$ for any ideals I, J of A .
- (3) $I \cap J \subset P$ implies $I \subset P$ or $J \subset P$ for any ideals I, J of A .
- (4) $\langle a \rangle \cap \langle b \rangle \subset P$ implies $a \in P$ or $b \in P$ for any a, b in A , where $\langle x \rangle$ denotes the principal ideal generated by x in A .

Theorem 7. Each ideal I of A is the intersection of all prime ideals containing I .

Corollary 8. The intersection of all prime ideals of A is $\{0\}$.

Theorem 9. If I is any ideal of A and D is a dual ideal of A such that $I \cap D = \emptyset$, then, there exists a prime ideal P of A such that $I \subset P$ and $P \cap D = \emptyset$.

Following Keimel [1], we denote by $\text{Spec } A$ the set of all prime ideals of A . For each subset M of A , let

$$S(M) = \{P \in \text{Spec } A \mid M \not\subset P\};$$

in particular, $S(a) = \{P \in \text{Spec } A \mid a \notin P\}$. Then we have the following from [3].

Lemma 10. The following hold:

- (a) $S(0) = \emptyset$, $S(A) = \text{Spec } A$.
- (b) $S(I \cap J) = S(I) \cap S(J)$ for all ideals I, J of A .
- (c) $S(\bigvee_{\alpha} I_{\alpha}) = \bigcup S(I_{\alpha})$ for any family $\{I_{\alpha}\}_{\alpha}$ of ideals of A .
- (d) $S(a) = S(\langle a \rangle)$ for any a in A .
- (e) $a \in I$ if and only if $S(a) \subset S(I)$.

Following Keimel [1], we introduce the following

Definition 11. We call *spectral topology* on $\text{Spec } A$ the topology whose open sets are the $S(I)$, where I runs through the set $I(A)$ of all ideals of A . We always endow $\text{Spec } A$ with this topology and call it the *Spectral space* of A (which is called the *Stone space* of A by Prabhakara Rao [3]).

Definition 12. If X is a subset of $\text{Spec } A$, then the topology on X induced by the spectral topology on $\text{Spec } A$ is called the *Spectral topology* on X .

The following theorems and their proofs are analogous to that of Keimel ([1], 1.24, 1.25).

Theorem 13. Let X be a subset of $\text{Spec } A$. For each subset M of A , let

$$S_X(M) = S(M) \cap X = \{P \in X \mid M \not\subseteq P\}.$$

Then, the sets $S_X(I)$, $I \in I(A)$, are the open sets of the spectral topology on X ; the sets $S_X(a)$, $a \in A$, form a basis of this topology and the following assertions hold:

- (a) $S_X(0) = \emptyset$, $S_X(A) = X$.
- (b) $S_X(I \cap J) = S_X(I) \cap S_X(J)$ for all ideals I, J of A .
- (c) $S_X(\bigvee_{\alpha} I_{\alpha}) = \bigcup_{\alpha} S_X(I_{\alpha})$ for each family $\{I_{\alpha}\}_{\alpha}$ of ideals of A .

Theorem 14. Let X be a subset of $\text{Spec } A$ and $\bar{X}$ be the closure of X in $\text{Spec } A$. Then the following assertions hold:

- (a) $\bar{X} = H(\bigcap X)$, the set of all prime ideals which contain $\bigcap X$.

(b) X is closed if and only if every prime ideal which contains $\bigcap X$ belongs to X .

(c) X is dense in $\text{Spec } A$ if and only if $\bigcap X = \{0\}$.

The following Theorems 15, 16 and 17 follow from (§1.3, [3]) of Prabhakara Rao.

Theorem 15. *The ideal lattice $I(A)$ of A is isomorphic with the lattice of all open sets of the (Spectral) space $\text{Spec } A$ of A .*

Theorem 16. *The join semilattice of all principal ideals of A is isomorphic with the join semilattice of all compact open sets of the space $\text{Spec } A$.*

Theorem 17. *For each ideal I of A , there exists a largest ideal $I^\perp$ of A such that $I \cap I^\perp = \{0\}$ and further, $I^\perp = \bigcap \{P \in \text{Spec } A \mid I \not\subseteq P\} = \bigcap_{S_X(I)} S_X(I) = \{x \in A \mid (x * 0) \cap (a * 0) = 0 \text{ for all } a \text{ in } I\}$, where X is any subspace of $\text{Spec } A$ such that $\bigcap X = \{0\}$.*

Definition 18. For every subset M of A , let $M^\perp$ denote the ideal $\langle M \rangle^\perp$, that is the largest ideal of A which has zero intersection with the ideal generated by M . In particular, let $a^\perp$ denote the largest ideal of A which has a zero intersection with the principal ideal $\langle a \rangle$ generated by a . $M^\perp$ is called the polar ideal of M and the ideal $M^{\perp\perp} = (M^\perp)^\perp$ is called the bipolar of M .

From [6], we recall the following

Theorem 19. *Any two congruence relations on A are permutable and the structure lattice $L(A)$ (i.e., the lattice of all congruence relations of*

A) is a distributive lattice.

Following U. Maddana Swamy [2], we introduce the following

Definition 20. A proper congruence relation ϕ on A is called (*meet*) *irreducible* if and only if, for any $\theta_1, \theta_2 \in L(A)$, $\theta_1 \cap \theta_2 \subset \phi$ implies that $\theta_1 \subset \phi$ or $\theta_2 \subset \phi$. Then, the set X of all irreducible congruence relations on A becomes a topological space by taking $\{X_\theta \mid \theta \in L(A)\}$ as open sets, where $X_\theta = \{\phi \in X \mid \theta \not\subset \phi\}$ for each $\theta \in L(A)$, which we call the (irreducible) spectrum of A .

Remark 21. In view of Theorem 2.7, [4], it is easy to observe that there is a 1-1 correspondence between prime ideals of A and the (meet) irreducible congruence relations of A . Further, it is clear that the (irreducible) spectrum X of A (defined above) is homeomorphic to the spectral space $\text{Spec } A$ of A . Hence, hereafter, we denote the spectrum X of A also by $\text{Spec } A$.

Now, we introduce the following

Definition 22. Let X and E be topological spaces and η a local homeomorphism from E onto X , that is a surjective function $\eta: E \rightarrow X$ with the property that for each $s \in E$, there are open neighbourhoods V and U of s in E and $\eta(s)$ in X respectively such that $\eta|_V$ is a homeomorphism from V onto U . Let

$$E \Delta E = \{(s, t) \in E \times E \mid \eta(s) = \eta(t)\} = \bigcup_{x \in X} \eta^{-1}(x) \times \eta^{-1}(x)$$

with the topology induced from the product space $E \times E$. Suppose that for each $x \in X$, the stalk $E_x = \eta^{-1}(x)$ is a representable autometrized algebra $E_x = (E_x, +, \leq, *)$ such that the operations

$(s, t) \mapsto s + t, (s, t) \mapsto s * t, (s, t) \mapsto s \cup t$ and $(s, t) \mapsto s \cap t$ from $E \Delta E$ into E are continuous. Then, the triple $F = \langle E, \eta, X \rangle$ is called a *sheaf of representable autometrized algebras over the base space* X . If each stalk $E_x, x \in X$, is a totally ordered representable autometrized algebra, then F is called *sheaf of totally ordered representable autometrized algebras*. We shall say that $F = \langle E, \eta, X \rangle$ is a T_2 -*sheaf* if the sheaf space E is a Hausdorff space.

Now, let $F = \langle E, \eta, X \rangle$ be a sheaf of representable autometrized algebras.

Definition 23. By a *section* of F we mean a continuous mapping $\sigma : X \rightarrow E$ such that $\eta \circ \sigma = \text{id}_X$ (= identity mapping on X). The set of all sections of F is denoted by $\Gamma(X, F)$.

Note 1. It can be verified that the function $x \mapsto o_x$, where o_x denotes the identity element (zero) of the stalk E_x , is a section of F and we denote this by 0 .

If σ and τ are two sections in F , then it is easy to see that $\{x \in X \mid \sigma(x) = \tau(x)\}$ is open and hence the *support* of each section σ , i.e.,

$$\text{supp } \sigma = \{x \in X \mid \sigma(x) \neq o_x\}$$

is closed in X . The set of all sections σ of F with compact supports is denoted by Γ_k^F or $\Gamma_k(X, F)$.

Note 2. For $\sigma, \tau \in \Gamma F$, we define $(\sigma + \tau)(x) = \sigma(x) + \tau(x)$, $(\sigma \cup \tau)(x) = \sigma(x) \cup \tau(x)$, $(\sigma \cap \tau)(x) = \sigma(x) \cap \tau(x)$ and $(\sigma * \tau)(x) = \sigma(x) * \tau(x)$ for all $x \in X$. Then, it can be verified in a routine way that $\Gamma F = (\Gamma F, +, \leq, *)$ is a representable autometrized algebra which is, in fact, a subdirect product of the stalks $E_x, x \in X$. Further, it is easy to verify that Γ_k^F

is an ideal of Γ_F .

Note 3. For each $x \in X$, we may define in a homomorphism $f_x: \Gamma F \rightarrow E_x$ by

$$f_x(\sigma) = \sigma(x) \quad \text{for each } \sigma \in \Gamma F.$$

We say that F is a *global sheaf* if f_x is surjective for each x in X .

It can be verified that every sheaf over a zero dimensional space is global.

The following theorem and its proof are analogous to that of Keimel ([1], Theorem 2.7).

Theorem 24. Let X be a topological space and $A = (A, +, \leq, *)$ be a representable autometrized algebra satisfying (R1). Let $\theta: X \rightarrow L(A)$ be a mapping such that for each $p \in A \times A$, $X(p) = \{x \in X \mid p \in \theta(x)\}$ is open in X . Let S be the disjoint union of all $A/\theta(x)$, $x \in X$, and define $\eta: S \rightarrow X$ by $\eta(s) = x$ for all $s \in A/\theta(x)$. For each $a \in A$, define $\hat{a}: X \rightarrow S$ by $x \mapsto \pi_x(a)$, where $\pi_x: A \rightarrow A/\theta(x)$ is the natural epimorphism. Let S be equipped with the largest topology that makes each $\hat{a}$, $a \in A$, continuous. Then, the following hold:

- i) The family $\{\hat{a}(U) \mid a \in A, U \text{ is open in } X\}$ forms a base for the topology on S .
- ii) The triple $F = \langle S, \eta, X \rangle$ is a global sheaf of representable autometrized algebras (satisfying (R1)) over X .
- iii) The mapping $a \mapsto \hat{a}$ is a homomorphism of A into ΓF , the representable autometrized algebra of all (global) sections of the sheaf F and this is a monomorphism if and only if $\bigcap_{x \in X} \theta(x) = \omega$ (= diagonal of A).
- iv) F is a T_2 -sheaf if and only if X is a T_2 -space and $X(p)$ is

closed also for all $p \in A \times A$.

By virtue of Theorems 19 and 24 and by applying Theorem 2, [2] of U. Maddana Swamy, we obtain the following theorem for a representable autometrized algebra $A = (A, +, \leq, *)$ satisfying (R1), and this theorem generalizes the corresponding Theorem 2.4.1 of Prabhakara Rao [3] for commutative DR ℓ -semigroups.

Theorem 25. *Let $A = (A, +, \leq, *)$ be a representable autometrized algebra satisfying (R1). Let Y be a dense subspace of $X = \text{Spec } A$. Let Z be any topological space and $\zeta : Y \rightarrow Z$ be a continuous map. Then, there exists a sheaf $F = \langle S, \eta, Z \rangle$ of homomorphic images of A over Z such that A is isomorphic with a subalgebra $\hat{A}$ of the algebra ΓF of all global sections of F . Further, if Y is a full and dense subspace of $\text{Spec } A$, then $\hat{A}$ contains all $\sigma \in \Gamma F$ such that $|\sigma, \hat{a}| = \{x \in Z \mid \sigma(x) \neq a(x)\}$ is compact for some $a \in A$, where $\hat{a} : Z \rightarrow S$ is defined by $\hat{a}(x) = \pi_x(a)$, where $\pi_x : A \rightarrow A/\theta(x)$ is the natural epimorphism (where $\hat{a} : Z \rightarrow S$ is defined as in Theorem 24).*

§ 2. Throughout this section, $A = (A, +, \leq, *)$ stands for a representable autometrized algebra satisfying the conditions:

$$(R1) \quad a * (a \cap b) + a \cap b = a \quad \text{for all } a, b \text{ in } A,$$

$$(R2) \quad \{a * (a \cup b)\} \cap \{b * (a \cup b)\} = 0 \quad \text{for all } a, b \text{ in } A.$$

We omit the proofs for the following theorems because the proofs given by Keimel [1] for ℓ -rings and f -rings can be extended in a routine manner with suitable modifications to a representable autometrized algebra $A = (A, +, \leq, *)$ satisfying (R1) and (R2).

Theorem 26. Let X be a dense subspace of $\text{Spec } A$ (spectral space of A). Let $P \in X$ and let O_P denote the germinal ideal (which is defined analogously as in Keimel [1]) associated with P (with respect to the space X and $\zeta = \text{id}_X$). Then

$$O_P = \bigcup \{a^\perp \mid a \notin P\}.$$

Remark 27. In Theorem 25, take $Y = X$, $Z = \text{Spec } A$ and $\zeta = \text{id}_X$. For each open subset U of Z , let $\theta_U = \bigcap \zeta^{-1}(U)$ and for each $z \in Z$, let $\theta_z = \bigcup \{\theta_U \mid U \text{ is an open neighbourhood of } z\}$. Then θ_z is a congruence relation of A and $Z((a, b)) = \{z \in Z \mid (a, b) \in \theta_z\}$ is open in Z . (All these details may be found in the proof of Theorem 2 of U. Maddana Swamy [2]). We observe that the germinal ideal O_P associated with $P \in X$ is the ideal corresponding to the congruence relation θ_P of A .

Theorem 28. For a proper ideal P of A , the following assertions are equivalent:

- (a) P is a prime ideal.
- (b) A/θ is totally ordered, where θ is the congruence relation of A corresponding to the ideal P .
- (c) For all a, b in A , $a \wedge b = 0$ implies $a \in P$ or $b \in P$.
- (d) For all $a, b \in A^+$, $a \wedge b \in P$ implies $a \in P$ or $b \in P$, where $A^+ = \{x \in A \mid x \geq 0\}$ is the positive cone of A .

Proof. (a) $\Rightarrow$ (b): Assume (a) and suppose that A/θ is not totally ordered. Then, there exist $a, b \in A$ such that $a * (a \cup b) \notin P$ and $b * (a \cup b) \notin P$. Hence, $\langle a * (a \cup b) \rangle \cap \langle b * (a \cup b) \rangle \not\subseteq P$ (by Theorem 6). So, there exists $c \in \langle a * (a \cup b) \rangle \cap \langle b * (a \cup b) \rangle$ such that $c \notin P$. That is, there exist positive integers m and n such that $c * 0 \leq m \{a * (a \cup b)\}$

and $c * 0 \leq n \{b * (a \cup b)\}$. Since $(A, +, \leq)$ is a commutative lattice ordered semigroup and $\{a * (a \cup b)\} \cap \{b * (a \cup b)\} = 0$ (by (R2)), it follows that $c = 0$ and hence $c \in P$, which is a contradiction to $c \notin P$. Hence, A/θ is totally ordered.

(b) $\Rightarrow$ (c): Assume (b) and let $a, b \in A$ such that $a \cap b = 0$. Then, we have $[a]^\theta \cap [b]^\theta = [0]^\theta$, which (by assumption) implies $[a]^\theta = [0]^\theta$ or $[b]^\theta = [0]^\theta$. (Here $[x]^\theta$ denotes the congruence class containing x relative to θ). This implies $a * 0 \in P$ or $b * 0 \in P$, which implies $a \in P$ or $b \in P$.

(c) $\Rightarrow$ (d): Assume (c). Let $a, b \in A$ such that $a \geq 0, b \geq 0$ and $a \cap b \in P$. By (R2), we have $\{a * (a \cup b)\} \cap \{b * (a \cup b)\} = 0$, i.e., $\{b * (a \cap b)\} \cap \{a * (a \cap b)\} = 0$ (since $x * (x \cup y) = y * (x \cap y)$ for all x, y in A). Hence, by assumption, it follows that $b * (a \cap b) \in P$ or $a * (a \cap b) \in P$. Since, $b = b * (a \cap b) + a \cap b$ and $a = a * (a \cap b) + a \cap b$ (by (R1)), it follows that $b \in P$ or $a \in P$.

(d) $\Rightarrow$ (a): Assume (d). Let $a, b \in A$ such that $a \notin P$ and $b \notin P$. Then, $a * 0 \notin P$ and $b * 0 \notin P$. So, by assumption, it follows that $(a * 0) \cap (b * 0) \notin P$. But, since $(a * 0) \cap (b * 0) \leq a * 0$ and $(a * 0) \cap (b * 0) \leq b * 0$, it follows that $(a * 0) \cap (b * 0) \in \langle a \rangle \cap \langle b \rangle$. Hence $\langle a \rangle \cap \langle b \rangle \notin P$. Hence, by Theorem 6 it follows that P is a prime ideal of A . This completes the proof.

Corollary 29. If P is a prime ideal of A , then the proper ideals of A containing P form a chain and they are all prime.

Corollary 30. The intersection of two principal ideals of A is a principal ideal of A ; more precisely,

$$\langle a \rangle \cap \langle b \rangle = \langle a \cap b \rangle \quad \text{for all } a, b \in A^+.$$

Theorem 31. *If P and Q are incomparable prime ideals of A , then P and Q have disjoint open neighbourhoods in $\text{Spec } A$.*

Proof. We have $P \not\subseteq Q$ and $Q \not\subseteq P$. So, there exist $a, b \in A^+$ such that $a \in Q \setminus P$ and $b \in P \setminus Q$. Let $u = a * (a \cap b)$ and $v = b * (a \cap b)$. Then, $u \notin P$ and $v \notin Q$; for, otherwise, since $a \cap b \in P \cap Q$, we have $a = u + a \cap b \in P$ or $b = v + a \cap b \in Q$, which is a contradiction. Hence $P \in S(u)$ and $Q \in S(v)$. But, since $u \cap v = 0$, we have $S(u) \cap S(v) = S(\langle u \rangle \cap \langle v \rangle)$ (by Lemma 10) $= S(\langle u \cap v \rangle)$ (by Corollary 30) $= S(0) = \emptyset$.

Corollary 32. *If X is a subspace of $\text{Spec } A$, and if any two prime ideals of A are incomparable, then X is a Hausdorff space.*

Definition 33. A prime ideal P of a representable autometrized algebra $A = (A, +, \leq, *)$ is called a *minimal prime ideal* if and only if it does not contain any prime ideal of A properly. A minimal irreducible congruence relation of A can be analogously defined. We denote the space of all minimal prime ideals of A endowed with the Spectral topology by $\pi A = \min \text{Spec } A$.

Lemma 34. *πA is dense in $\text{Spec } A$.*

Proof. We observe that the intersection of a chain of prime ideals of A is still a prime ideal of A . Hence each prime ideal of A contains a minimal one by Zorn's lemma. Consequently, the intersection of the family of all minimal prime ideals of A is $\{0\}$. Hence, by Theorem 14, it follows that πA is dense in $\text{Spec } A$.

Following Keimel [1], we introduce the following

Definition 35. A subset F of A^+ is called an *f-set* if and only if F satisfies

- (i) $0 \notin F$, and
- (ii) $a, b \in F$ implies $a \cap b \in F$.

If P is a prime ideal of A , then $A^+ \setminus P$ is an *f-set* (by Theorem 28).

Each *f-set* is contained in a maximal one (by Zorn's lemma). Maximal *f-sets* can be characterized as follows.

Lemma 36. An *f-set* F of A^+ is a maximal *f-set* if and only if for each x in A with $x * 0 \notin F$, there is an $a \in F$ such that $(x * 0) \cap a = 0$.

Proof. Routine as in [1].

Lemma 37. If F is a maximal *f-set*, then

$$P = \bigcup \{a^\perp \mid a \in F\} = \{x \in A \mid x * 0 \notin F\}$$

is a maximal prime ideal of A with $F = A^+ \setminus P$.

Proof. Routine as in [1].

Corollary 38. If P is a maximal prime ideal of A , then $A^+ \setminus P$ is a maximal *f-set*.

Proof. Routine as in [1].

Corollary 39. A minimal prime ideal P of A has the following properties:

- (i) For all a in A one and only one of $a \in P$ and $a^\perp \subset P$ will happen.

(ii) $a \in P$ implies $a^{\perp\perp} \subset P$.

Proof. Routine as in [1].

Theorem 40. *Let P be a prime ideal of A . Then, the germinal ideal associated with P is*

$$O_P = \bigcup \{a^\perp \mid a \notin P\} = \bigcap \{Q \in \pi A \mid Q \subset P\}.$$

Proof. Routine as in [1].

Corollary 41. *If P is a minimal prime ideal of A , then*

$$P = O_P = \bigcup \{a^\perp \mid a \notin P\}.$$

Theorem 42. *The sets of the form $S_{\pi A}(a) = \{P \in \pi A \mid a \notin P\}$, $a \in A$, are closed and open in πA and they form a basis of the topology on πA : In particular, πA is a zero dimensional Hausdorff space.*

Proof. Routine as in [1].

Now, we have the main theorem of the paper, which generalizes the corresponding result (Theorem 2.4.16, [3]) of Prabhakara Rao for representable DR ℓ -semigroups (A representable DR ℓ -semigroup is a commutative DR ℓ -semigroup $B = (B, +, \leq, -)$ satisfying the condition: $(a - b) \cap (b - a) \leq 0$ for all a, b in B).

Theorem 43. *Let $A = (A, +, \leq, *)$ be a representable autometrized algebra satisfying (R1) and (R2). Then, the space πA of all minimal prime ideals of A is a zero dimensional Hausdorff space. There is a T_2 -sheaf $F(A, \pi A)$ of totally ordered representable autometrized algebras over πA such that A is isomorphic to a subalgebra $\hat{A}$ of the representable autometrized algebra of sections of $F(A, \pi A)$.*

Proof. By Theorem 42, it is clear that πA is a zero dimensional Hausdorff space. By Lemma 34, πA is a dense subspace of $\text{Spec } A$. Taking $Y = \pi A$, $Z = \pi A$ and $\zeta = \text{id}_{\pi A}$ in Theorem 25, we obtain a sheaf $F(A, \pi A)$ of homomorphic images of A over πA such that A is isomorphic with a subalgebra $\hat{A}$ of the representable autometrized algebra ΓF of all global sections of $F(A, \pi A)$. The stalks A/θ_P of $F(A, \pi A)$ are totally ordered since $O_P = P$ for each $P \in \pi A$ by Corollary 41 (θ_P is the congruence relation of A corresponding to the ideal P of A). Further, the set

$$h(a) = \{P \in \pi A \mid a \in O_P\} = \{P \in \pi A \mid a \in P\} = \pi A \setminus S_{\pi A}(a)$$

is closed for all in A . Thus, the sheaf space of $F(A, \pi A)$ is Hausdorff (by Theorem 24) and all the stalks are totally ordered. Hence, it follows that $F(A, \pi A)$ is a T_2 -sheaf of totally ordered representable autometrized algebras over πA . This completes the proof.

The author wishes to thank Dr. K. L. N. Swamy for his valuable guidance and encouragement in preparing this paper.

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