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On BCI-Algebras

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In my Note [1], the notion of BCI-algebra was introduced, and some elementary properties were proved. A BCI-algebra has not been studied until now. In this Note, we discuss BCI-algebras.

First we recall the definition of a BCI-algebra. Let X be an abstract algebra of type $\langle 2, 0 \rangle$ with a binary operation $*$ and a constant 0 . Then X is a *BCI-algebra*, if the following conditions satisfied for all $x, y, z \in X$:

$$(1) \quad ((x * y) * (x * z)) * (z * y) = 0.$$

$$(2) \quad (x * (x * y)) * y = 0.$$

$$(3) \quad x * x = 0.$$

$$(4) \quad x * y = y * x = 0 \text{ implies } x = y.$$

$$(5) \quad x * 0 = 0 \text{ implies } x = 0.$$

If (5) is replaced by $0 \leq x$, then the algebra is called a *BCK-algebra*.

Let X be a BCK-algebra, and suppose $x * 0 = 0$ ($x \in X$). Then since $0 \leq x$, we have $0 * x = 0$, and hence (4) implies $x = 0$. Therefore we have

Theorem 1. *Any BCK-algebra is a BCI-algebra.*

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From (1), $z \leq y$ implies $x * y \leq x * z$. Hence if $x \leq y$ and $y \leq z$, then $x \leq z$. Therefore a BCI-algebra is a partially ordered set with respect to $\leq$. Hence, the element 0 is a minimal element in a BCI-algebra X . If 0 is the least element in X , then X is a BCK-algebra (see [2]).

If $x * y \leq x$ holds for any x, y of a BCI-algebra X , then $0 * y \leq 0$ for any y of X . By a basic property of a BCI-algebra (see [1]), we have $0 = 0 * 0 \leq y$. Hence X is a BCK-algebra. Thus, if $x * y \leq x$ in a BCI-algebra X , then X is a BCK-algebra.

Example 1. There is a BCI-algebra which is not a BCK-algebra. Let $X = \{0, 1, a\}$. We define arbitrary operation $*$ as shown in Table 1.

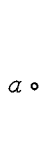


Figure 1

$*$	0	1	a
0	0	0	a
1	1	0	a
a	a	a	0

Table 1

Then X is a BCI-algebra with respect to $*$, and the subset $\{0, 1\}$ is a BCK-algebra. Generally, let X be a BCK-algebra, and let a be an ideal element which is

not contained in X . We define

$$x * a = a * x = a, a * a = 0$$

for any $x (\neq a) \in X$. Then $\{X, a\}$ is a BCI-algebra, but not a BCK-algebra.

To verify (1), it is sufficient to only consider the formula (1) that the left side contains a . Consider $(x * a) * (x * z)$. Then we must consider some cases. Let $x = a$. Then

$$(a * a) * (a * z) = \begin{cases} 0, & \text{if } z = a, \\ a, & \text{if } z \neq a. \end{cases}$$

Hence, (1) holds. Let $x \neq a$. Then

$$(x * a) * (x * z) = a * (x * z) = \begin{cases} 0, & \text{if } z = a, \\ a, & \text{if } z \neq a. \end{cases}$$

Hence (1) holds in this case. Consider $(x * y) * (x * a)$. Let $y = a$.

Then

$$(x * y) * (x * a) = 0.$$

Let $y \neq a$. Then

$$(x * y) * (x * a) = \begin{cases} a, & \text{if } x = a, \\ a, & \text{if } x \neq a. \end{cases}$$

Hence, as is easily seen, (1) holds. Consider $(a * y) * (a * z)$. If $y, z \neq a$, then $(a * y) * (a * z) = 0$. Let $y = a$. Then

$$(a * a) * (a * z) = \begin{cases} 0, & \text{if } z = a, \\ a, & \text{if } z \neq a. \end{cases}$$

Let $y \neq a, z = a$. Then

$$(a * y) * (a * a) = (a * y) * 0 = a.$$

Therefore, it follows that (1) holds in all cases.

We verify the formula (2). If $x, y \in X$, then (2) holds. If $x = a$, then

$$a * (a * y) = \begin{cases} 0, & \text{if } y \neq a, \\ a, & \text{if } y = a. \end{cases}$$

Hence, if $x = a$, then (2) holds. If $x \neq a$ and $y = a$, then

$$x * (x * a) = x * a = a.$$

Hence (2) holds in each case.

The verification of other formulas are trivial. Hence (X, a) is a BCI-algebra, but not a BCK-algebra.

We may consider the category of BCI-algebras which is denoted by BCI. Of course, morphisms are given by homomorphisms. By a homomorphism $f: X \rightarrow Y$

is defined by $f(x * y) = f(x) * f(y)$ for any $x, y \in X$. Then the category BCK is a subcategory of BCI.

Let X be a proper BCI-algebra, and B the set of all elements x such that $0 \leq x$. Then by (1), we have $0 * y \leq x * y$. Since $y \in B$, $0 * y = 0$. Hence $x * y \in B$. This means that B is a maximal BCK-subalgebra of X , therefore any BCI-algebra contain a maximal BCK-algebra.

Let $a \in X - B$. If $x * a \in B$ for some $x \in B$. Then $x * (x * a) \in B$. Hence $0 \leq x * (x * a) \leq a$ implies $0 \leq a$, which contradicts $a \notin B$. Thus for any $x \in B$ and $a \notin B$, $x * a$ is not contained in B .

In general, $x * y \leq x$ does not hold in a BCI-algebra X . But $(x * y) * z = (x * z) * y$ holds in X . Hence, for any $a \in X - B$, we have $0 = (a * a) * 0 = (a * 0) * a$. On the other hand, $a * (a * 0) \leq 0$. Thus $a * (a * 0) = 0$. By (4), we have $a * 0 = a$.

From (1), for any $x \in B$, $a \in X - B$, we have

$$(a * x) * (a * 0) \leq 0 * x = 0.$$

Hence from $a * 0 = a$, it follows that $a * x \leq a$. If $a * x \in B$, then $a \in B$, which is a contradiction. Hence for any $x \in B$, $a \in X - B$, $a * x$ is contained in $X - B$.

Theorem 1. *The maximal BCK-subalgebra B in a BCI-algebra is an ideal, i.e. $x * y, y \in B$ implies $x \in B$.*

Proof. Let $x * y, y \in B$. If $x \notin B$, by the above remark, it follows that $x * y \in X - B$. This is a contradiction.

Next we shall prove the fundamental relation

$$(6) \quad (x * y) * z = (x * z) * y.$$

The proof is quite similar to the case of a BCK-algebra. For completeness,

we shall give the proof in detail.

From (1), it follows that $z * y$ implies $x * y \leq x * z$. Hence, by (1), we have

$$(7) \quad u * (z * y) \leq u * ((x * y) * (x * z)).$$

We substitute $x * u$ for x , $x * z$ for z , and $((x * u) * y) * (z * u)$ for u in (7). Then

$$\begin{aligned} & (((x * u) * y) * (z * u)) * ((x * z) * y) \\ & \leq (((x * u) * y) * (z * u)) * (((x * u) * y) * ((x * u) * (x * z))). \end{aligned}$$

By (7), the right side of the above inequality equals to 0. Hence, by (5), we have

$$(((x * u) * y) * (z * u)) * ((x * z) * y) = 0,$$

which means

$$(8) \quad ((x * u) * y) * (z * u) \leq (x * z) * y.$$

We substitute z for u , and $x * y$ for z in (8), then we have

$$((x * z) * y) * ((x * y) * z) \leq (x * (x * y)) * y.$$

By (2), the right side of the above inequality equals to 0. Thus we have

$$(x * z) * y \leq (x * y) * z,$$

which implies (6).

Conversely, assume (1), (3), (4), (5), (6). Then

$$(x * (x * y)) * y = (x * y) * (x * y) = 0,$$

which implies $x * (x * y) \leq y$.

S. Tanaka introduced an important and fundamental class of a BCK-algebra which is said to be commutative (see [2]): A BCK-algebra X is *commutative* if $x * (x * y) = y * (y * x)$ for any $x, y \in X$.

Let X be a BCI-algebra in which $x * (x * y) = y * (y * x)$ holds for any x, y .

Consider an element $a \in X - B$. Then $0 * (0 * a) = a * (a * 0) = a * a = 0$. Hence $0 \leq 0 * a$. As is already shown, $0 * a \in X - B$, which is a contradiction. Thus $X = B$. So we conclude

Theorem 2. A BCI-algebra in which

$$x * (x * y) = y * (y * x)$$

holds for any x, y is a BCK-algebra.

Theorem 3. If

$$(9) \quad (x * y) * y = x * y$$

holds for any x, y of a BCI-algebra, then it is a BCK-algebra.

Proof. Let $x = y$ in (9). Then $(x * x) * x = x * x = 0$. Hence $0 * x = 0$ for any x . Thus the algebra is a BCK-algebra.

Theorem 4. If

$$x * (y * x) = x$$

holds for any x, y of a BCI-algebra X , then the algebra X is a BCK-algebra.

Proof. Let $x = 0$. Then

$$0 * (y * 0) = 0.$$

Since $y * 0 = y$, we have $0 * y = 0$ for any y . Hence $X - B$ is empty. Thus X is a BCK-algebra.

References

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- [2] K. Iséki and S. Tanaka: An introduction to the theory of BCK-algebras, Math. Japonica, 23 (1978), 1-26.