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Kasahara, Shouro

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A Fixed Point Theorem of Meir-Keeler Type

By Shouro KASAHARA*

The purpose of this note is to present a fixed point theorem for a continuous self-mapping of a complete metric space which satisfies a contractive condition of Meir-Keeler type. A generalization of a theorem of Wong [4] is also stated.

1. As a slight generalization of Theorem 1 of [3], we shall prove the following theorem which is the main result of this note.

Theorem 1. *Let* f *be a continuous mapping of a complete metric space* (X, d) *into itself which carries a subset* M *of* X *into* M. *Suppose that each point of* M *is regular*[1] *for* f *and that there exists a positive integer valued function* k *on* M *satisfying the following condition*:

(*) *For any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $x \in M$ *and* $\varepsilon \leq D(x) < \varepsilon + \delta$ *imply* $D(f^{k(x)}(x)) < \varepsilon$, *where* $D(x)$ *denotes the diameter of the orbit* $O_f(x) = \{f^n(x) \mid n \in \omega\}$[2] *of* x *under* f.

Then for any $x_0 \in M$, *the sequence* $\{f^n(x_0)\}_{n\in\omega}$ *converges to a fixed point*

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* Department of Mathematics, Kobe University, Nada, Kobe, 657, Japan.

1) See [2] or [3].

2) The symbol ω denotes the set of all nonnegative integers.

of f.

Proof. Put $x_n = f^n(x_0)$ for each $n \in \omega$. Since $D(x_{n+1}) \le D(x_n)$ for every $n \in \omega$, the sequence $\{D(x_n)\}_{n\in\omega}$ converges to some $\varepsilon \ge 0$. Suppose $\varepsilon > 0$. Then by (*), there exists a $\delta > 0$ such that $x \in M$ and $\varepsilon \le D(x) < \varepsilon + \delta$ imply $D(f^{k(x)}(x)) < \varepsilon$. Since $\varepsilon \le D(x_n) < \varepsilon + \delta$ for some $n \in \omega$, it follows that $D(x_{n+k(x_n)}) = D(f^{k(x_n)}(x_n)) < \varepsilon$, which is absurd. Thus we have $\varepsilon = 0$ which means that $S = \{x_n\}_{n\in\omega}$ is a Cauchy sequence. Hence, S converges to some $a \in X$. Therefore the subsequence $\{f(x_n)\}_{n\in\omega}$ of S converges to $f(a)$. Thus $f(a) = a$.

The following extends Corollary 1[3] of [3].

Corollary. *Let* f *be a continuous mapping of a nonempty complete metric space* (X, d) *into itself. Suppose that each point of* X *is regular for* f *and that there exists a positive integer valued function* k *on* X *satisfying the following condition*:

(**) *For any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $x, y \in X$ *and* $\varepsilon \le D(x, y) < \varepsilon + \delta$ *imply* $D(f^{k(x)+k(y)}(x), f^{k(x)+k(y)}(y)) < \varepsilon$, *where* $D(x, y)$ *denotes the diameter of* $O_f(x) \cup O_f(y)$.

Then f *has a unique fixed point, and for any* $x \in X$, $\{f^n(x)\}_{n\in\omega}$ *converges to the fixed point.*

Proof. In view of (**), for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $x \in M$ and $\varepsilon \le D(x) < \varepsilon + \delta$ imply $D(f^{2k(x)}(x)) < \varepsilon$. Hence Theorem 1 shows

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[3] The first sentence of the corollary should read "*Let* f *be a continuous mapping of a nonempty complete metric space* (X, d) *into itself*", and "M" on the page should read "X".

that for any $x \in X$, the sequence $\{f^n(x)\}_{n\in\omega}$ converges to a fixed point of f. Since X is nonempty, this guarantees the existence of a fixed point of f. If $x, y \in X$ are distinct fixed points of f, then by (**), there exists a $\delta > 0$ such that $u, v \in X$ and

$$d(x, y) \le D(u, v) < d(x, y) + \delta$$

imply

$$D(f^{k(u)+k(v)}(u), f^{k(u)+k(v)}(v)) < d(x, y),$$

which yields a contradiction since

$$D(x, y) = D(f^{k(x)+k(y)}(x), f^{k(x)+k(y)}(y)) = d(x, y).$$

This completes the proof.

2. The following theorem, motivated by lemmas of Hegedüs and Szilágyi [1], extends Theorem 1 of Wong [4] and clarifies the position of the Meir-Keeler type contractive conditions such as (*) and (**). The set of all nonnegative real numbers will be denoted by $\mathbb{R}_+$.

Theorem 2. *The following conditions on a subset* Q *of* $\mathbb{R}_+ \times \mathbb{R}_+$ *are equivalent*:

(1) $(x, y) \in Q$ *implies* $y \le x$, *and for every* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $(x, y) \in Q$ *and* $\varepsilon \le x < \varepsilon + \delta$ *imply* $y < \varepsilon$.

(2) *For every* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $(x, y) \in Q$ *and* $x < \varepsilon + \delta$ *imply* $y < \varepsilon$.

(3) *There exists a nondecreasing extended real valued function* w *on* $\mathbb{R}_+$ *such that*

$1°$ $w(t) > t$ *for all* $t > 0$, *and*

$2°$ $w(y) \le x$ *for all* $(x, y) \in Q$.

(4) *There exists an extended real valued function* w *on* $\mathbb{R}_+$ *which is*

lower semicontinuous from the right on $\mathbb{R}_+ \setminus \{0\}$ *and satisfies* 1° *and* 2°.

Proof. The implications (1) $\Rightarrow$(2) and (3) $\Rightarrow$(4) are obvious. To prove the implication (2) $\Rightarrow$(3), put $w(0) = 0$, and let $w(\varepsilon)$ denote, for each $\varepsilon > 0$, the supremum of the set $M(\varepsilon)$ of all $\delta > 0$ such that $(x, y) \in Q$ and $x < \delta$ imply $y < \varepsilon$. It is easy to see that w is a nondecreasing extended real valued function on $\mathbb{R}_+$. If $\varepsilon > 0$, then (2) shows that $\varepsilon + \delta \in M(\varepsilon)$ for some $\delta > 0$ and so $w(\varepsilon) > \varepsilon$. Suppose $w(y) > x$ for some $(x, y) \in Q$. Then there exists a $\delta \in M(y)$ such that $\delta > x$. Hence $(u, v) \in Q$ and $u < \delta$ imply $v < y$. But this is absurd since $(x, y) \in Q$ and $x < \delta$. Therefore $w(y) \le x$ for all $(x, y) \in Q$. It remains only to prove the implication (4) $\Rightarrow$(1). Suppose (4) holds, and let $\varepsilon > 0$. Then there exists a δ' > 0 such that

$$\frac{\varepsilon + w(\varepsilon)}{2} < w(t)$$

whenever $\varepsilon \le t < \varepsilon + \delta'$, since w is lower semicontinuous from the right at ε. Put $\delta = \min\{\delta', (w(\varepsilon) - \varepsilon)/2\}$. If $\varepsilon \le x < \varepsilon + \delta$ and $y \ge \varepsilon$ for some $(x, y) \in Q$, then $\varepsilon \le y < w(y) \le x < \varepsilon + \delta \le \varepsilon + \delta'$, and hence we have

$$\frac{\varepsilon + w(\varepsilon)}{2} < w(y) \le x \le \varepsilon + \delta \le \varepsilon + \frac{w(\varepsilon) - \varepsilon}{2} = \frac{\varepsilon + w(\varepsilon)}{2},$$

a contradiction. Thus $(x, y) \in Q$ and $\varepsilon \le x < \varepsilon + \delta$ imply $y \ge \varepsilon$. Finally let (x, y) be in Q. Then $y > 0$ implies $y < w(y) \le x$, and $y = 0$ implies $y = 0 \le x$. This completes the proof.

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