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A Surgical View of Alexander Invariants of Links

By Yasutaka NAKANISHI*

1. Introduction

An *n-components link* $L = K_1 \cup \cdots \cup K_n$ is an ordered collection of n disjoint simple closed oriented curves K_i's in a three dimensional sphere S^3. Two links are said to have the *same link type* if there is an orientation-preserving homeomorphism from S^3 to itself, which maps one link into the other, preserving the orientation and order of the components.

An *Alexander matrix* $M_L(t_1, \cdots, t_n)$ of L is a presentation matrix of the first homology group $H_1(\tilde{X}_a)$ as a Λ-module, where $\tilde{X}_a$ means the universal abelian covering space of the exterior X of L and Λ means the integral group-ring $ZH_1(X)$; we can see $ZH_1(X) = \Lambda$ as the Laurent polynomial ring $Z[t_1, t_1^{-1}, \cdots, t_n, t_n^{-1}]$ where t_i is always taken to be represented by the meridian of K_i. An Alexander polynomial $\Delta_L(t_1,\cdots,t_n)$ of L is a generator of the smallest principal ideal containing the order ideal of $M_L(t_1, \cdots, t_n)$ ([3], [7]).

In this paper, we will give a characterization of Alexander matrices

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of 2-components links (Theorems 4 and 5) and prove that the Torres's conditions are sufficient for 2-components links with the linking number 0 and 1 (Theorem 7).

For a manifold X, ∂X means the boundary of X and $^{\circ}X$ means the interior of X. For a manifold-pair (X, Y), $N(Y)$ means the tubular neighbourhood of a submanifold Y in X.

Throughout this paper, we work in piecewise linear category.

2. 2-components links

Proposition 1. *A* $(2, 2k)$-*torus link may be expressed as the union of two ball-pairs* $(B^3, B^1 \cup B^1) \underset{\theta}{\cup} (B'^3, B'^1 \cup B'^1)$ *shown in Figure* 1, *where* $\theta : \partial B^3 \to \partial B'^3$ *is an orientation-preserving homeomorphism with* $\theta(y) = y'$ *and* $\theta(z) = z'$ *and* $\theta(d_i) = d_i'$ $(i = 1, 2, 3, 4)$.

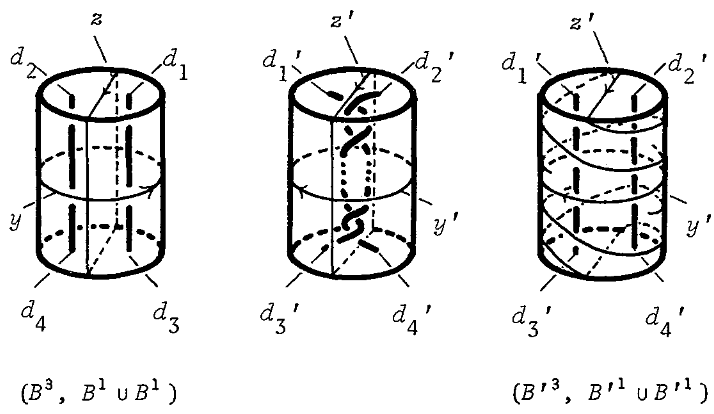


Figure 1

In Figure 1, the points d_i (resp. d_i') are $\partial(B^1 \cup B^1)$ (resp. $\partial(B'^1 \cup B'^1)$) and the oriented circles y, z (resp. y', z') are on ∂B^3 (resp. $\partial B'^3$), and the middle figure shows the k-full twisted $(B'^3, B'^1 \cup B'^1)$. The observation on Figure 1 will suggest the proof of Proposition 1.

A link L is said to be *transformed* to a link L' iff there is a finite sequence of links $L = L_0, \cdots, L_n = L'$ satisfying the followings:

(1) L_{i+1} is ambient isotopic to L_i,

or (2) L_{i+1} is obtained from L_i by replacing subarcs of the same component in a local 3-ball as in Figure 2.

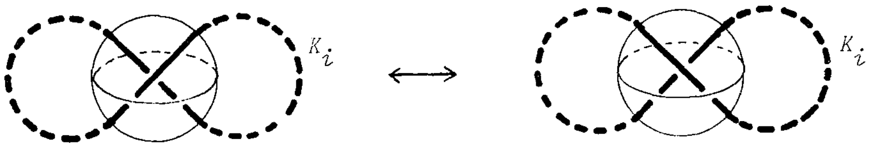


Figure 2

We can easily show the following fact.

Proposition 2a. *For any 2-components link with the linking number* k, *a* $(2, 2k)$*-torus link can be transformed to the given one.*

Proposition 2a can be stated in a surgical description ([8], [9], [11], [12]) as follows.

Proposition 2b. *Let* $L = K_1 \cup K_2$ *be a 2-components link with the linking number* k *in* S^3 *and let* L_0 *be a* $(2, 2k)$*-torus link in* S^3. *Then there exist* ν *disjoint solid tori* $T_1, \cdots, T_\nu$ *in* $S^3 - L_0$ *and a homeomorphism* ϕ *from* $S^3 - {}^{o}T_1 \cup \cdots \cup {}^{o}T_\nu$ *to itself such that*

(1) $\phi(L_0) = L$,

(2) $T_1 \cup \cdots \cup T_\nu$ *is a trivial link,*

(3) $lk(T_i, \phi^{-1}(K_j)) = lk(T_i, K_j) = 0$ *for all* i, j,

(4) $\phi(\partial T_i) = \partial T_i$ *and* $lk(\mu_i', T_i) = 1$ *where* $\mu_i \subset \partial T_i$ *is a meridian of* T_i *and* $\mu_i' = \phi^{-1}(\mu_i)$,

(5) T_i*'s are in* B'^3 *when* S^3 *is separated as in Proposition* 1.

In Proposition 2b and hereafter, for convenience, we use the same symbol for a solid torus and its core unless confusion.

First we consider the universal abelian covering space $\tilde{X}_{a0}$ of the exterior of L_0. Let D (resp. D', F, F') be the closure of $B^3 - N(B^1 \cup B^1)$ (resp. $B'^3 - N(B'^1 \cup B'^1)$, $\partial B^3 - \partial N(B^1 \cup B^1)$, $\partial B'^3 - \partial N(B'^1 \cup B'^1)$). Then $D \underset{\theta}{\cup} D'$ is homeomorphic to the exterior of L_0. We denote by $\tilde{D}$ (resp. $\tilde{D}'$, $\tilde{F}$, $\tilde{F}'$) the lift of D (resp. D', F, F') in $\tilde{X}_{a0}$. In Figure 3, (i) shows D' (resp. D) and the oriented circles σ, ω (resp. s, z) on F' (resp. F), (ii) shows a fundamental region of $\tilde{D}'$ (resp. $\tilde{D}$), and (iii) shows $\tilde{D}'$ (resp. $\tilde{D}$) which is visualized in R^3 where $\tilde{\sigma}$ or $\tilde{\omega}$ (resp. $\tilde{s}$ or $\tilde{z}$)

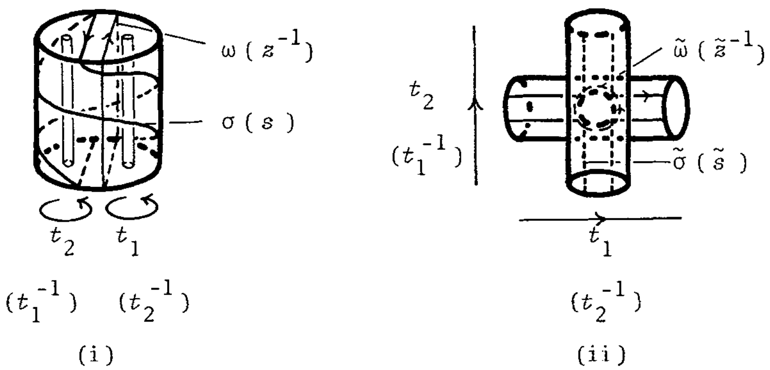


Figure 3

is a component of the lift of σ or ω (resp. s or z) respectively.

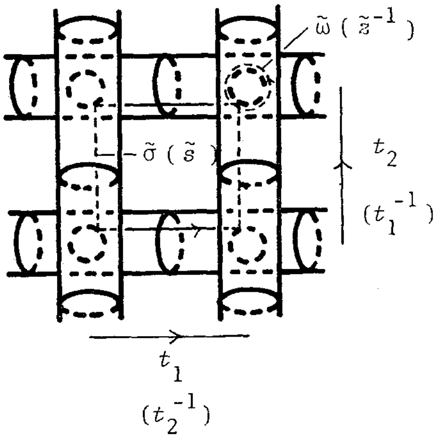
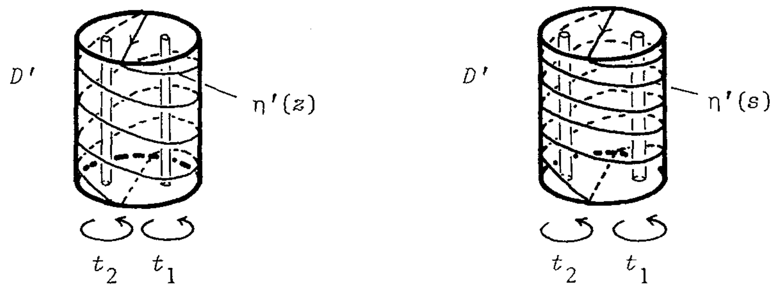


Figure 3 (iii)

Let η be an inclusion map from F to D and let η' be a composite map of a restriction map of θ to F and an inclusion map from F' to D'. We denote by $\tilde{\theta}$ (resp. $\tilde{\eta}$, $\tilde{\eta}'$) a lift of θ (resp. η, η'). Then $\tilde{\eta}'(\tilde{s})$ and $\tilde{\eta}'(\tilde{z})$ are homologous to

$$\frac{(t_1t_2)^{k+1}-1}{t_1t_2-1}\tilde{\sigma}+\frac{(t_1t_2)^{k}-1}{t_1t_2-1}\tilde{\omega} \quad \text{and} \quad \frac{(t_1t_2)^{k}-1}{t_1t_2-1}\tilde{\sigma}+\frac{(t_1t_2)^{k-1}-1}{t_1t_2-1}\tilde{\omega}$$

respectively. In the case $k = 2$, they appear as in Figure 4, (i) shows $\eta'(z)$ on F', (ii) shows $\eta'(s)$ on F', and (iii) shows $\tilde{\eta}'(\tilde{z})$, $\tilde{\eta}'(\tilde{s})$ on $\tilde{F}'$.



(i) (ii)

Figure 4

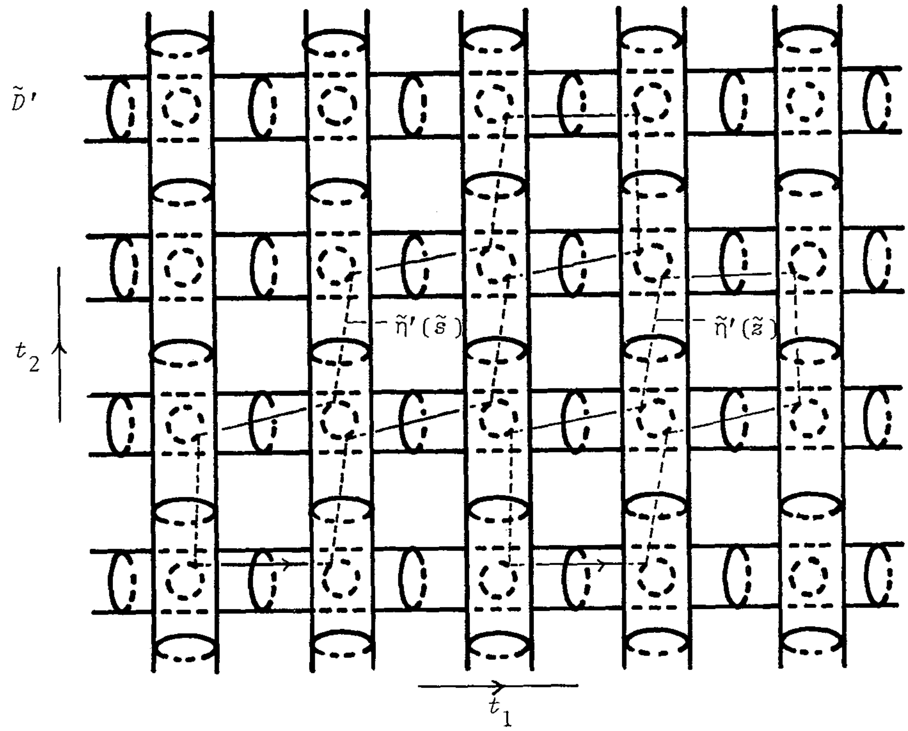


Figure 4 (iii)

We denote by $<\cdots:\cdots>$ a presentation of a homology group as a $\Lambda = Z[t_1, t_1^{-1}, t_2, t_2^{-1}]$-module. Then we have

$$H_1(\tilde{F}) \cong <\tilde{z}, \tilde{s}: \quad >,$$

$$H_1(\tilde{D}) \cong <\tilde{s}: \quad >,$$

$$H_1(\tilde{D}') \cong <\tilde{\sigma}: \quad >.$$

We have the following exact sequence by the Mayer-Vietoris theorm.

$$H_1(\tilde{F}) \xrightarrow[\tilde{\eta}_* \oplus \tilde{\eta}'_*]{} H_1(\tilde{D}) \oplus H_1(\tilde{D}') \longrightarrow H_1(\tilde{X}_{a0}) \longrightarrow 0.$$

(The maps $\tilde{\eta}_*$, $\tilde{\eta}'_*$ are homomorphisms induced by $\tilde{\eta}$, $\tilde{\eta}'$ respectively.)

Since $Im(\tilde{\eta}_* \oplus \tilde{\eta}'_*)$ is generated by $\tilde{s} \oplus \tilde{\eta}'(\tilde{s})$ and $0 \oplus \tilde{\eta}'(\tilde{z})$, we have

$$H_1(\tilde{X}_{a0}) \cong H_1(\tilde{D}')/< \tilde{\eta}'(\tilde{z}): \quad >.$$

We have two universal abelian covering spaces; $p : \tilde{X}_a \to S^3 - {}^{o}N(L)$, which we wish to study, and $p_0 : \tilde{X}_{a0} \to S^3 - {}^{o}N(L_0)$, which is understood in the above. We know that T_i's in Proposition 2b are lifted to solid tori in either covering space from Proposition 2b (3). Let $\tilde{T}_i \subset \tilde{X}_{a0}$ be a fixed component of the lift of T_i, so $p_0^{-1}(T_i) = \bigcup_{u,v} t_1^u t_2^v \tilde{T}_i$ in $\tilde{X}_{a0}$ and let $\tilde{\mu}_i'$ be the component of the lift of μ_i' in $\partial\tilde{T}_i$. Then $\tilde{X}_a$ may be obtained from $\tilde{X}_{a0}$ by removing the disjoint solid tori $t_1^u t_2^v \tilde{T}_i$ ($i = 1, \cdots, \nu$; $u, v = 0, \pm 1, \cdots$) and sewing back, by ψ, each so that its meridian coincides to $t_1^u t_2^v \tilde{\mu}_i'$.

The linking number of two circles in $\tilde{D}'$ means the linking number in R^3 in which $\tilde{D}'$ is embedded as in Figure 4 (iii).

Let us calculate $H_1(\tilde{X}_a)$. For each $i = 1, \cdots, \nu$, we choose a 1-cycle $\tilde{\alpha}_i$ in $\tilde{X}_{a0} - \bigcup_{u,v,i} t_1^u t_2^v \, {}^{o}\tilde{T}_i$ dual to $\tilde{T}_i$ in the sence that

$$lk(\tilde{\alpha}_i, t_1^u t_2^v \tilde{T}_j) = \begin{cases} 1 & i = j \text{ and } u = v = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\tilde{E}' = \tilde{D}' - \bigcup_{u,v,i} t_1^u t_2^v \, {}^{o}\tilde{T}_i$. Since $\tilde{E}'$ is invariant under covering translation of $\tilde{D}'$, $H_1(\tilde{E}')$ is a Λ-module. From the exact sequence of Λ-modules

$$0 = H_2(\tilde{D}') \longrightarrow H_2(\tilde{D}', \tilde{E}') \longrightarrow H_1(\tilde{E}') \longrightarrow H_1(\tilde{D}') \longrightarrow H_1(\tilde{D}', \tilde{E}') = 0$$

and the fact that, by excision, $H_2(\tilde{D}', \tilde{E}')$ is the free Λ-module on generators represented by ν meridian disks, we have that $H_1(\tilde{E}')$ is a free

Λ-module on generators $\tilde{\sigma}, \tilde{\alpha}_1, \cdots, \tilde{\alpha}_\nu$. Since $\tilde{X}_a \cong (\tilde{E}' \underset{\psi}{\cup} \bigcup_{u,v,i} t_1^u t_2^v \tilde{T}_i) \underset{\tilde{\theta}}{\cup} \tilde{D}$, we have $H_1(\tilde{X}_a) \cong < \tilde{\sigma}, \tilde{\alpha}_1, \cdots, \tilde{\alpha}_\nu : \tilde{\eta}'(\tilde{z}) = \tilde{\mu}_1' = \cdots = \tilde{\mu}_\nu' = 0 >$.

Example. For the link $L = 9^2_{29}$ ([12]) as in Figure 5 (i), we have

$$H_1(\tilde{X}_a) \cong < \tilde{\sigma}, \tilde{\alpha}_1, \tilde{\alpha}_2 : \tilde{\eta}'(\tilde{z}) = \tilde{\mu}_1' = \tilde{\mu}_2' = 0 >.$$

In Figure 5, we obtain (i) from (ii) by +1-surgeries along T_1 and T_2 where the meridians of T_1 and T_2 in (i) are the images of μ_1' and μ_2' respectively. (ii) is deformed to (iii), in which the area surrounded by dotted line is B'^3. (iv) shows a fundamental region of $\tilde{D}'$. Simplifying (iv) to (v), we have (vi) which shows $\tilde{D}'$ and the lifts of T_1 and T_2. From (vi), we can see that

$$\begin{aligned}
\tilde{\eta}'(\tilde{z}) &= (1 + t_1 t_2)\tilde{\sigma} + (1 - t_1)(1 - t_2)\tilde{\alpha}_2, \\
\tilde{\mu}_1' &= (-t_1 + 1 - t_1^{-1})\tilde{\alpha}_1 + (-t_1^{-1} t_2^{-1} + t_2^{-1})\tilde{\alpha}_2, \\
\tilde{\mu}_2' &= \tilde{\sigma} + (-t_1 t_2 + t_2)\tilde{\alpha}_1 + (-t_1 + 1 - t_1^{-1})\tilde{\alpha}_2.
\end{aligned}$$

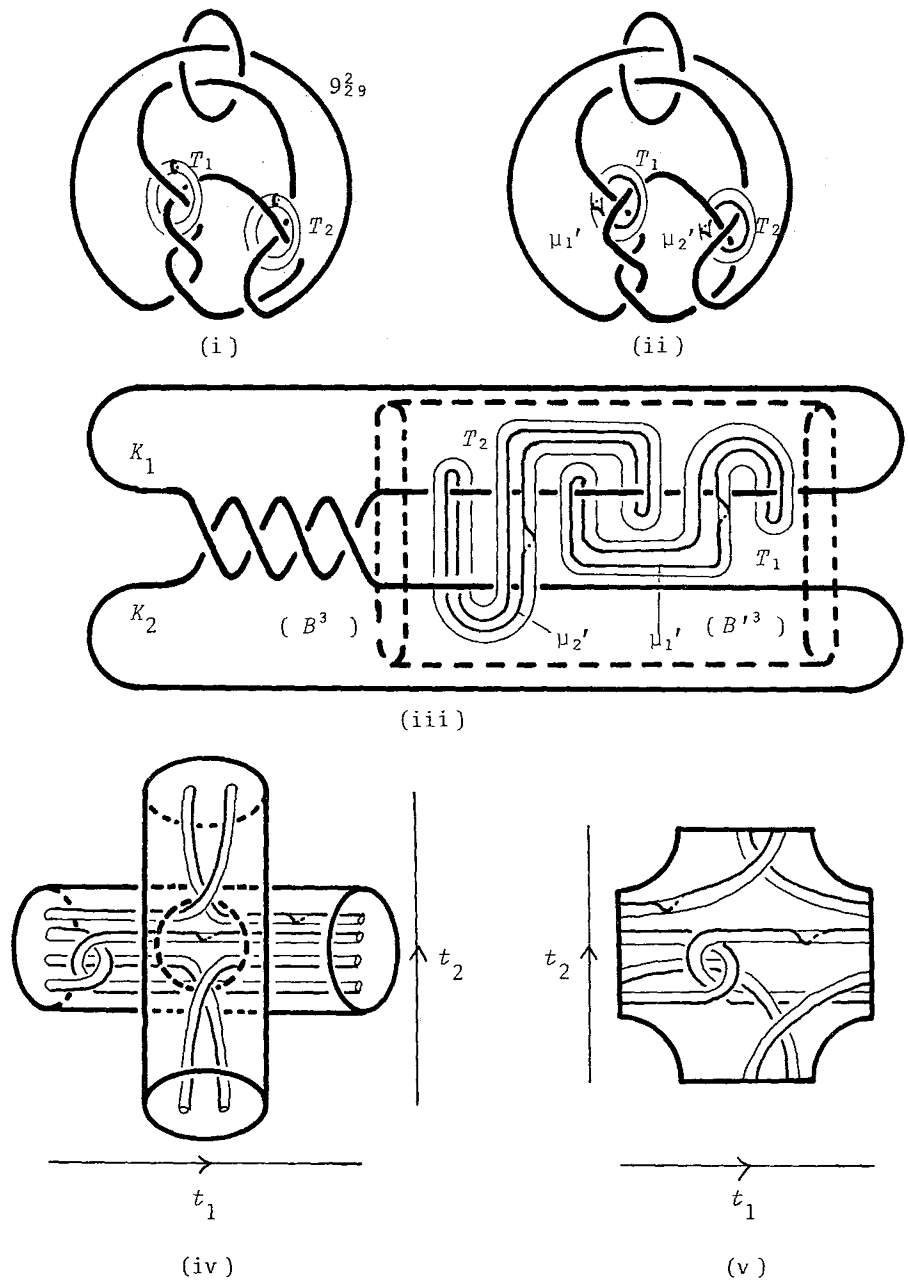
Thus we have an Alexander matrix $M_L = M_L(t_1, t_2)$ of L;

$$M_L = \begin{pmatrix} 1 + t_1 t_2 & 0 & (1 - t_1)(1 - t_2) \\ 0 & -t_1 + 1 - t_1^{-1} & -t_1^{-1} t_2^{-1} + t_2^{-1} \\ 1 & -t_1 t_2 + t_2 & -t_1 + 1 - t_1^{-1} \end{pmatrix}.$$

and an Alexander polynomial $\Delta_L = \Delta_L(t_1, t_2) = det\, M_L$;

$$\Delta_L = 1 - 2t_1 + 3t_1^2 - 3t_1^3 + 2t_1^4 + 2t_1 t_2 - 3t_1^2 t_2 + 3t_1^3 t_2 - 2t_1^4 t_2 + t_1^5 t_2.$$

Remark. The Alexander polynomial of 9^2_{29} in [12] is incorrect. We will note errata in [12] within sight in Appendix.



9^2_{29}
T_1
T_2
(i)
T_1
μ_1'
μ_2'
T_2
(ii)
K_1
K_2
T_2
T_1
(B^3)
μ_2'
μ_1'
(B'^3)
(iii)
t_2
t_1
(iv)
t_2
t_1
(v)

Figure 5

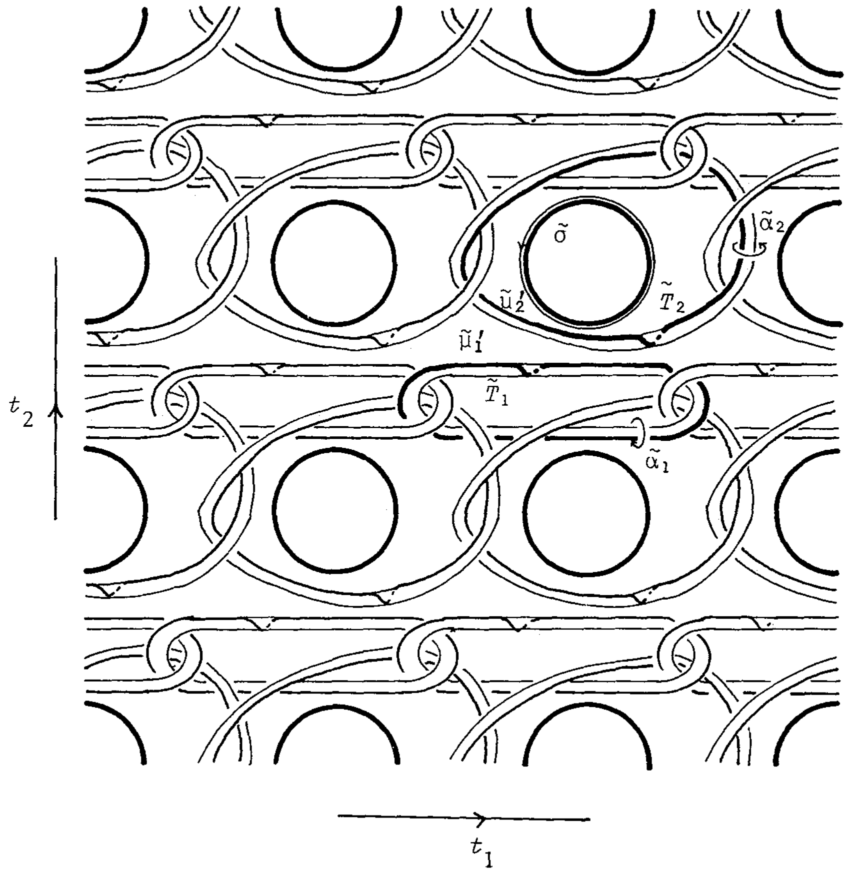


Figure 5 (vi)

Now we consider coefficients of Alexander matrices. We present $\tilde{\mu}_i{}'$, $\tilde{\eta}'(\tilde{z})$ as linear-combinations of $\tilde{\sigma}, \tilde{\alpha}_1, \cdots, \tilde{\alpha}_\nu$ as follows.

(a) $\tilde{\mu}_i{}' = f_i(t_1, t_2)\tilde{\sigma} + \sum_{1\le j\le \nu} m_{ij}(t_1, t_2)\tilde{\alpha}_j$,

(b) $\tilde{\eta}'(\tilde{z}) = \dfrac{(t_1t_2)^k - 1}{t_1t_2 - 1}\tilde{\sigma} + \dfrac{(t_1t_2)^{k-1} - 1}{t_1t_2 - 1}\tilde{\omega}$,

where $\tilde{\omega} = \sum_{1\le i\le \nu} (\sum_{u,v} lk(\tilde{\omega}, t_1^u t_2^v \tilde{T}_i) t_1^u t_2^v \tilde{\alpha}_i)$.

Pushing $\tilde{\sigma}$ off $\tilde{F}'$ into $\tilde{D}'$, we denote it by $\hat{\sigma}$. Then we have

$$lk(\tilde{\omega},\ t_1^u t_2^v \hat{\sigma}) = (\delta_{u,0} - \delta_{u,1})(\delta_{v,0} - \delta_{v,1})$$

where δ_{ij} means Kronecker's delta. Since $\tilde{T}_i$ is homologous to $f_i(t_1,\ t_2)\hat{\sigma}$ in $\tilde{D}'$, we have

$$\sum_{u,v} lk(\tilde{\omega},\ t_1^u t_2^v \tilde{T}_i)\, t_1^u t_2^v = f_i(t_1^{-1},\ t_2^{-1})(1 - t_1)(1 - t_2).$$

Hence we have

$$\text{(b}'\text{)}\quad \tilde{\eta}'(\tilde{z}) = \frac{(t_1 t_2)^k - 1}{t_1 t_2 - 1}\tilde{\sigma} + \frac{(t_1 t_2)^{k+1} - 1}{t_1 t_2 - 1}(1-t_1)(1-t_2)\sum_{1\le i\le \nu} f_i(t_1^{-1},\ t_2^{-1})\tilde{\alpha}_i.$$

We will discuss the Alexander matrix M_L.

Lemma 3. *Between the coefficients of relations* (a), *we have*

(1) $m_{ij}(t_1,\ t_2) = m_{ji}(t_1^{-1},\ t_2^{-1})$,

(2) $|m_{ij}(1,\ 1)| = \delta_{ij}$.

Proof. (1) is a consequence of the following equations.

$$\begin{aligned} lk(\tilde{\mu}_i',\ t_1^u t_2^v \tilde{T}_j) &= lk(\tilde{T}_i,\ t_1^u t_2^v \tilde{T}_j) \quad \text{(except when } i = j \text{ and } u = v = 0) \\ &= lk(t_1^{-u} t_2^{-v} \tilde{T}_i,\ \tilde{T}_j) \\ &= lk(\tilde{T}_j,\ t_1^{-u} t_2^{-v} \tilde{T}_i) \\ &= lk(\tilde{\mu}_j',\ t_1^{-u} t_2^{-v} \tilde{T}_i). \end{aligned}$$

(2) is a consequence of the following equations.

$$lk(T_i,\ T_j) = \sum_{u,v} lk(\tilde{T}_i,\ t_1^u t_2^v \tilde{T}_j) = \sum_{u,v} lk(t_1^u t_2^v \tilde{T}_i,\ \tilde{T}_j).$$

Theorem 4. *Let* $L = K_1 \cup K_2$ *be a 2-components link with* $lk(K_1, K_2) = k$. *Then an Alexander matrix of* L *is equivalent to the matrix of the form:*

$$\begin{pmatrix} \dfrac{(t_1t_2)^k - 1}{t_1t_2 - 1} & g_1(t_1, t_2) & \cdots & g_\nu(t_1, t_2) \\ f_1(t_1, t_2) & m_{11}(t_1, t_2) & \cdots & m_{1\nu}(t_1, t_2) \\ \vdots & \vdots & & \vdots \\ f_\nu(t_1, t_2) & m_{\nu 1}(t_1, t_2) & \cdots & m_{\nu\nu}(t_1, t_2) \end{pmatrix}$$

satisfying the following properties

(1) $m_{ij}(t_1, t_2) = m_{ji}(t_1^{-1}, t_2^{-1})$,

(2) $|m_{ij}(1, 1)| = \delta_{ij}$,

(3) $g_i(t_1, t_2) = \dfrac{(t_1t_2)^{k-1} - 1}{t_1t_2 - 1}(1 - t_1)(1 - t_2)f_i(t_1^{-1}, t_2^{-1})$.

Considering carefully the above argument, we can see that the converse to Theorem 4 is valid. That is

Theorem 5. *For any matrix of the form in Theorem 4 satisfying the properties* (1), (2) *and* (3) *there exists a 2-components link* L *in* S^3 *whose Alexander matrix is equivalent to the given one.*

By Theorem 4, we have

$$\det M_L(t_1, t_2) = \frac{(t_1t_2)^k - 1}{t_1t_2 - 1}\det M' + \frac{(t_1t_2)^{k-1} - 1}{t_1t_2 - 1}\det M''(1 - t_1)(1 - t_2),$$

where $M' = \begin{pmatrix} m_{11}(t_1, t_2) & \cdots & m_{1\nu}(t_1, t_2) \\ \vdots & & \vdots \\ m_{\nu 1}(t_1, t_2) & \cdots & m_{\nu\nu}(t_1, t_2) \end{pmatrix}$,

$$M'' = \begin{pmatrix} 0 & f_1(t_1^{-1}, t_2^{-1}) & \cdots & f_\nu(t_1^{-1}, t_2^{-1}) \\ f_1(t_1, t_2) & & & \\ \vdots & & M' & \\ f_\nu(t_1, t_2) & & & \end{pmatrix}.$$

From $m_{ij}(t_1, t_2) = m_{ji}(t_1^{-1}, t_2^{-1})$, we have

$\det M_L(t_1, t_2) = t_1^{c_1} t_2^{c_2} \det M_L(t_1^{-1}, t_2^{-1})$ for some $c_i \equiv k+1 \pmod 2$.

It can be seen that $M'|_{t_2=1}$ and $M'|_{t_1=1}$ are Alexander matrices of K_1 and K_2 respectively. Hence we have the Torres's conditions [14, 15] for 2-components links. That is

Corollary 6. *Let $\Delta_L(t_1, t_2)$ be an Alexander polynomial of a link $L = K_1 \cup K_2$ in S^3 with $k = lk(K_1, K_2)$. Then we have*

(1) $\Delta_L(t_1, 1) \doteq \dfrac{t_1^k - 1}{t_1 - 1} \Delta_{K_1}(t_1)$,

(2) $\Delta_L(1, t_2) \doteq \dfrac{t_2^k - 1}{t_2 - 1} \Delta_{K_2}(t_2)$,

(3) $\Delta_L(t_1, t_2) = t_1^{c_1} t_2^{c_2} \Delta_L(t_1^{-1}, t_2^{-1})$ *for some* $c_i \equiv k+1 \pmod 2$,

where $\Delta_{K_i}(t_i)$ is an Alexander polynomial of K_i and $\doteq$ means "equal up to units".

Question. *Given any polynomial $\Delta(t_1, t_2)$ with 2-variables t_1, t_2 which satisfies*

(1) $\Delta(t_1, 1) \doteq \dfrac{t_1^k - 1}{t_1 - 1}\Delta_1(t_1)$ *with* $\Delta_1(t_1) \doteq \Delta_1(t_1^{-1})$, $|\Delta_1(1)| = 1$,

(2) $\Delta(1, t_2) \doteq \dfrac{t_2^k - 1}{t_2 - 1}\Delta_2(t_2)$ *with* $\Delta_2(t_2) \doteq \Delta_2(t_2^{-1})$, $|\Delta_2(1)| = 1$,

(3) $\Delta(t_1, t_2) = t_1^{c_1} t_2^{c_2} \Delta(t_1^{-1}, t_2^{-1})$ *for some* $c_i \equiv k + 1 \pmod 2$,

(4) $|\Delta(1, 1)| = k$.

Does there exist a 2-components link L *in* S^3 *whose Alexander polynomial is equal to* $\Delta(t_1, t_2)$ *up to units?*

[9], [13] give a proof for the case $k = 1$, [6] gives a counterexample for the case $k = 6$. In Theorem 7, we give a partial answer.

Theorem 7. *The answer for the case* $k = 0, 1$ *in Question is "Yes".*

Proof. In the case $k = 1$, $\Delta(t_1, t_2) = det \begin{pmatrix} 1 & 0 \\ 0 & \Delta(t_1, t_2) \end{pmatrix}$ implies the existence of a required link by Theorem 5.

In the case $k = 0$, we may assume

$\Delta(t_1, t_2) = (1 - t_1)(1 - t_2)\Delta'(t_1, t_2)$ with $\Delta'(t_1, t_2) = \Delta'(t_1^{-1}, t_2^{-1})$.

We may suppse

$$\Delta'(t_1, t_2) = \sum_{\substack{-\kappa\le i\le\kappa \\ 1\le j\le\lambda}} a_{ij}\Big(t_1^i t_2^j + \sum_{0\le v\le j-1} t_2^v\Big)\Big(t_1^{-i} t_2^{-j} + \sum_{0\le v\le j-1} t_2^{-v}\Big)$$

$$+ \sum_{-\kappa\le i\le\kappa} a_{i0}\Big(\sum_{0\le u\le i} t_1^u\Big)\Big(\sum_{0\le u\le i} t_1^{-u}\Big) + a_{00}.$$

We denote

$${}_{ij}f_l = {}_{ij}f_l(t_1, t_2) = \begin{cases} t_1^i t_2^j + \sum_{0\le v\le j-1} t_2^v & (j \ne 0), \\ \sum_{0\le u\le i} t_1^u & (j = 0), \end{cases}$$

$${}_{ij}g_l = {}_{ij}g_l(t_1, t_2) = (1 - t_1)(1 - t_2)\,{}_{ij}f_l(t_1^{-1}, t_2^{-1}),$$

$$
\left(\begin{array}{c|c|c|c|c|c|c|c}
0 & {}_{00}g_1 \cdots {}_{00}g_{b_{00}} & {}_{10}g_1 \cdots {}_{10}g_{b_{10}} & \cdots & {}_{\kappa 0}g_1 \cdots {}_{\kappa 0}g_{b_{\kappa 0}} & {}_{-\kappa 1}g_1 \cdots {}_{-\kappa 1}g_{b_{-\kappa 1}} & \cdots & {}_{\kappa\lambda}g_1 \cdots {}_{\kappa\lambda}g_{b_{\kappa\lambda}} \\
\hline
\begin{matrix} {}_{00}f_1 \\ \vdots \\ {}_{00}f_{b_{00}} \end{matrix} & \begin{matrix} -s_{00} & & \\ & \ddots & \\ & & -s_{00} \end{matrix} & & & & & & \\
\hline
\begin{matrix} {}_{10}f_1 \\ \vdots \\ {}_{10}f_{b_{10}} \end{matrix} & & \begin{matrix} -s_{10} & & \\ & \ddots & \\ & & -s_{10} \end{matrix} & & & \bigcirc & & \\
\hline
\vdots & & & \ddots & & & & \\
\hline
\begin{matrix} {}_{\kappa 0}f_1 \\ \vdots \\ {}_{\kappa 0}f_{b_{\kappa 0}} \end{matrix} & & & & \begin{matrix} -s_{\kappa 0} & & \\ & \ddots & \\ & & -s_{\kappa 0} \end{matrix} & & & \\
\hline
\begin{matrix} {}_{-\kappa 1}f_1 \\ \vdots \\ {}_{-\kappa 1}f_{b_{-\kappa 1}} \end{matrix} & & \bigcirc & & & \begin{matrix} -s_{-\kappa 1} & & \\ & \ddots & \\ & & -s_{-\kappa 1} \end{matrix} & & \\
\hline
\vdots & & & & & & \ddots & \\
\hline
\begin{matrix} {}_{\kappa\lambda}f_1 \\ \vdots \\ {}_{\kappa\lambda}f_{b_{\kappa\lambda}} \end{matrix} & & & & & & & \begin{matrix} -s_{\kappa\lambda} & & \\ & \ddots & \\ & & -s_{\kappa\lambda} \end{matrix}
\end{array}\right)
$$

The matrix P

s_{ij} = the sign of a_{ij},

$b_{ij} = |a_{ij}|$.

Consider the preceding matrix P, it is easily seen that $det\,P$ becomes equal to $\Delta(t_1, t_2)$. That implies the existence of a required link.

Note. We will give a surgical description of a link corresponding to the matrix P. Consider ball-pairs ${}_{ij}D_w$ ($-\kappa \leq i \leq \kappa$, $0 \leq j \leq \lambda$, $1 \leq w \leq b_{ij}$) which include the solid tori ${}_{ij}T_w$ with $-s_{ij}$-surgery respectively as in Figure 6. We attach the bottom face of ${}_{ij}D_w$ to the top face of ${}_{ij}D_{w+1}$ for $1 \leq w \leq b_{ij}-1$, and get ${}_{ij}D = {}_{ij}D_1 \cup \cdots \cup {}_{ij}D_{b_{ij}}$. Further we attach the bottom face of ${}_{ij}D$ to the top face of ${}_{i+1\,j}D$ for $0 \leq i \leq \kappa-1$, $j = 0$ or $-\kappa \leq i \leq \kappa-1$, $1 \leq j \leq \lambda$, and the bottom face of ${}_{\kappa j}D$ to the top face of ${}_{-\kappa\, j+1}D$ for $0 \leq j \leq \lambda-1$, and we obtained the surgery description $D' = {}_{00}D \cup \cdots \cup {}_{\kappa\lambda}D$ of a required link L corresponding to P. Now the linking number of L is 0 and the Alexander matrix becomes equivalent to P.

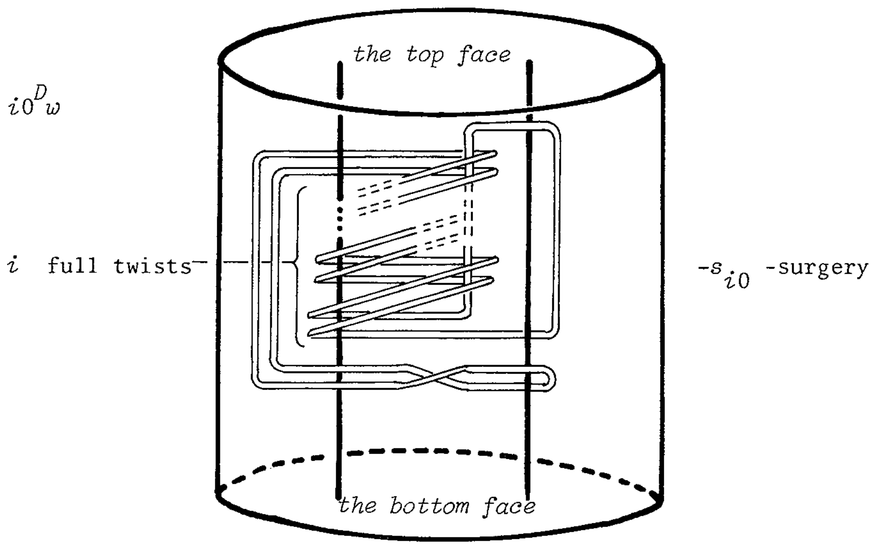


Figure 6 (i)

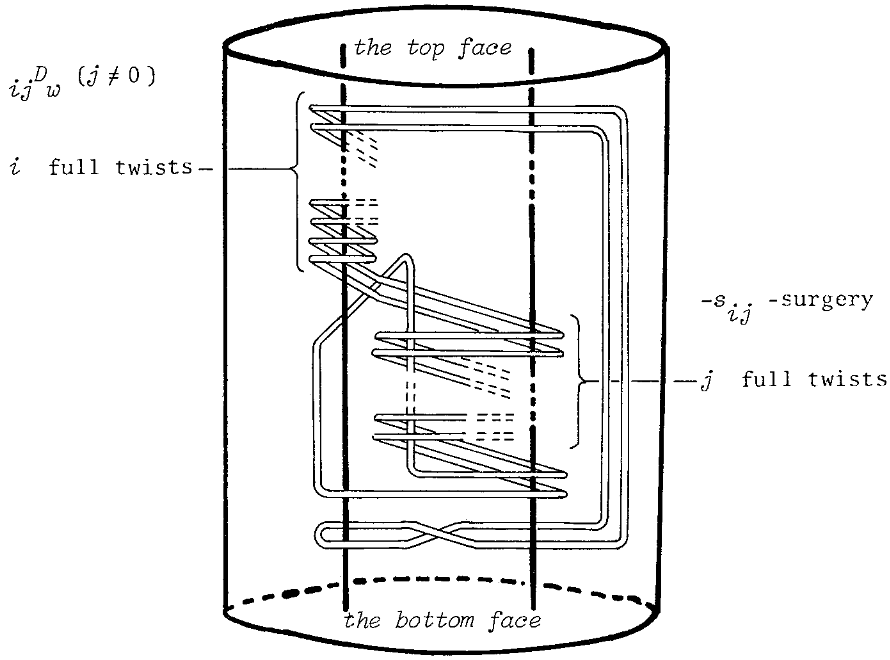


Figure 6 (ii)

3. n-components links ($n \geq 3$)

Proposition 8. *For any n-components link, there exists a closed pure n-braid* ([10]) *which can be transformed to the given one.*

Let $\tilde{X}_a$ be the universal abelian covering space of the exterior of L, then we have the following presentation of $H_1(\tilde{X}_a)$ as a $\Lambda = Z[t_1, t_1^{-1}, \cdots, t_n, t_n^{-1}]$ -module in a similar way to Section 2.

$H_1(\tilde{X}_a) \cong < \tilde{\sigma}_{ij}, \tilde{\alpha}_h$ $(1 \leq i,j \leq n,\ 1 \leq h \leq \nu)$:

$$\tilde{\rho}_{ijk},\ \tilde{\eta}'(\tilde{z}_l),\ \tilde{\mu}_h'\ (1 \leq i,j,k \leq n,\ 1 \leq l \leq n\text{-}1,\ 1 \leq h \leq \nu) >,$$

where $\tilde{\rho}_{ijk} = (1 - t_k)\tilde{\sigma}_{ij} + (1 - t_i)\tilde{\sigma}_{jk} + (1 - t_j)\tilde{\sigma}_{ki}$ with $\tilde{\sigma}_{ij} = -\tilde{\sigma}_{ji}$, and $\tilde{\eta}'(\tilde{z}_l)$ and $\tilde{\mu}_h'$ are Λ-linear combinations which depend on a surgical description. Thus we have an Alexander matrix with $\binom{\nu}{3} + \nu - 1 + n$ rows and $\binom{\nu}{2} + \nu$ columns.

Generally, it is hard to characterize Alexander matrices of links, but this surgical method will give us a method of characterization for the indivisual class of links which can be transformed to the same closed pure braid. For example, we note the following without proof.

Theorem 9. *If an n-components link L can be transformed to a trivial link, then Alexander matrices of L are characterized by the following;*

$$\begin{array}{cc}
 & \begin{array}{ccccccc|ccc} \cdots & \tilde{\sigma}_{uv} & \tilde{\sigma}_{vw} & \tilde{\sigma}_{wu} & \cdots & & & \tilde{\alpha}_1 & \cdots & \tilde{\alpha}_\nu \end{array} \\
\begin{array}{c} \vdots \\ \tilde{\rho}_{uvw} \\ \vdots \\ \tilde{\eta}'(\tilde{z}_1) \\ \vdots \\ \tilde{\eta}'(\tilde{z}_{n-1}) \\ \tilde{\mu}_1' \\ \vdots \\ \tilde{\mu}_\nu' \end{array} &
\left(\begin{array}{ccccc|ccc}
 & & & & & & & \\
0\cdots0 & 1-t_w & 1-t_u & 1-t_v & 0\cdots0 & & 0 & \\
 & & & & & & & \\ \hline
 & & & & & {}_1g_1 & \cdots & {}_1g_\nu \\
 & & 0 & & & \vdots & & \vdots \\
 & & & & & {}_{n-1}g_1 & \cdots & {}_{n-1}g_\nu \\ \hline
\cdots & {}_{uv}f_1 & {}_{vw}f_1 & {}_{wu}f_1 & \cdots & m_{11} & \cdots & m_{1\nu} \\
 & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
\cdots & {}_{uv}f_\nu & {}_{vw}f_\nu & {}_{wu}f_\nu & \cdots & m_{\nu 1} & \cdots & m_{\nu\nu}
\end{array}\right)
\end{array}$$

satisfying the properties

(1) $m_{ij}(t_1, \cdots, t_n) = m_{ji}(t_1^{-1}, \cdots, t_n^{-1})$,

(2) $|m_{ij}(1, \cdots, 1)| = \delta_{ij}$,

(3) ${}_ig_k(t_1, \cdots, t_n) = \sum\limits_{1 \le j \le n} {}_{ij}f_k(t_1^{-1}, \cdots, t_n^{-1})(1 - t_i^{-1})(1 - t_j^{-1})$

with ${}_{ij}f_k = -{}_{ji}f_k$.

Appendix. The following is a list of errata of Alexander polynomials in [12] within sight. For convenience of display, Alexander polynomials have been abbreviated as in [12].

$$9^2_{29} \left[\begin{array}{cccccc} 1 & -2 & 3 & -3 & 2 & 0 \\ 0 & 2 & -3 & 3 & -2 & 1 \end{array}\right. \qquad 9^2_{57} \left[\begin{array}{cccc} 1 & -2 & 1 & -1 \\ -1 & 1 & -2 & 1 \end{array}\right.$$

$$9^2_{55} \left[\begin{array}{cccc} -1 & 2 & -2 & 1 \\ 1 & -2 & 2 & -1 \end{array}\right. \qquad 9^2_{59} \left[\begin{array}{cccccc} 0 & 0 & 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 & 0 & 0 \end{array}\right.$$

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