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Realization of Lattice Ordered Autometrized Algebras

By B. V. SUBBA RAO

Introduction. Ribenboim [3] introduced the notion of a realization of a commutative ℓ-group by totally ordered commutative ℓ-groups and has shown that any commutative ℓ-group has a zero-dimensional realization. Later, Banaschewski [1] introduced the notion of a realization of an arbitrary ℓ-group by totally ordered ℓ-groups and has shown that an ℓ-group admits a zero-dimensional realization if and only if each ultrafilter in its positive part is a normal filter. Later, Rama Rao [2] has shown that a DRℓ-semigroup $A = (A, +, \leq, -)$ satisfying the condition

$$a - b \cup c = (a - b) \cap (a - c) \qquad \text{for all } a, b, c \in A,$$

admits a zero-dimensional realization if each ultrafilter in its positive part is normal. This is a partial extension of Banaschewski's result, since ℓ-groups are also DRℓ-semigroups.

The interest in Rama Rao's result is that it is applicable to Boolean algebras, since any DRℓ-semigroup is a common abstraction of Boolean algebras and ℓ-groups.

Swamy [7] introduced the notion of a lattice ordered autometrized algebra in connection with obtaining a unified theory of autometrized algebras.

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The calss of lattice ordered autometrized algebras include commutative DRℓ-semigroups [7] and commutative ℓ-groups as special cases. We extend the results of Banaschewski and Rama Rao to a special subclass of lattice ordered autometrized algebras, namely, the class of representable autometrized algebras.

In [4], we have introduced the concept of a representable autometrized algebra and observed that the class of representable autometrized algebras is wider than the class of commutative DRℓ-semigroups. We have also proved that a lattice ordered autometrized algebra $A = (A, +, \leq, *)$ is a subdirect product of totally ordered representable autometrized algebras if and only if A is a representable autometrized algebra and $(a * (a \cup b)) \cap (b * (a \cup b)) = 0$ for all a, b in A.

In §1 of this paper, we introduce the notions of a filter and a prime filter in the positive part P of a representable autometrized algebra $A = (A, +, \leq, *)$ satisfying the condition

(i) $(a * (a \cup b)) \cap (b * (a \cup b)) = 0$ for all a, b in A,

and make a study of homomorphisms, particularly epimorphisms of A into totally ordered representable autometrized algebras satisfying (i). In §2, we introduce the notion of a realization of a representable autometrized algebra A by totally ordered representable autometrized algebras and prove that every representable autometrized algebra satisfying (i) admits a zero-dimensional realization. This result generalizes the corresponding result of Ribenboim [3] for commutative ℓ-groups.

§ 1. First, we recall the following definition from [4].

Definition 1. A lattice ordered autometrized algebra $A = (A, +, \leq, *)$

is called a *representable autometrized algebra* if and only if

(1.1) A is a semiregular autometrized algebra, and

(1.2) For every a in A, all the mappings $x \mapsto a + x$, $x \mapsto a \cup x$, $x \mapsto a \cap x$ and $x \mapsto a * x$ are contractions (a mapping $f : A \to A$ is called a *contraction mapping* with respect to a metric $*$ on A if and only if $f(x) * f(y) \leq x * y$ for all x, y in A).

The definitions of an autometrized algebra and a semiregular autometrized algebra may be found in [8], and the definition of a lattice ordered autometrized algebra may be found in [7].

Definition 2 ([8]). A nonempty subset I of an autometrized algebra $A = (A, +, \leq, *)$ is called an *ideal* if and only if

(I1) $a \in I$, $b \in I$ imply $a + b \in I$, and

(I2) $a \in I$, $b \in A$ and $b * 0 \leq a * 0$ imply $b \in X$.

Definition 3 ([8]). An equivalence relation θ on an autometrized algebra $A = (A, +, \leq, *)$ is called a *congruence relation* if and only if

(C1) $a \equiv b(\theta)$, $c \equiv d(\theta)$ imply that $a + c \equiv b + d(\theta)$ for any a, b, c, d in A,

(C2) $a \equiv b(\theta)$, $c \equiv d(\theta)$ imply that $a * c \equiv b * d(\theta)$ for any a, b, c, d in A, and

(C3) $a \equiv b(\theta)$ and $x * y \leq a * b$ imply that $x \equiv y(\theta)$ for any a, b, x, y in A.

In [4], we have observed that the lattice of ideals of a representable autometrized algebra A is isomorphic with the lattice of congruence relations of A. We also proved (Theorem 2.8, [4]) that if θ is a congruence

relation on a representable autometrized algebra A, then, the quotient A/θ (the set of all congruence classes) is also a representable autometrized algebra.

Throughout this paper, $A = (A, +, \leq, *)$ stands for a representable autometrized algebra satisfying the condition

(1.3) $(a * (a \cup b)) \cap (b * (a \cup b)) = 0$ for all a, b in A.

It is easy to observe that if θ is a congruence relation on A, then, the representable autometrized algebra A/θ, also, satisfies (1.3) (Lemma 2.12.3, [4]).

In this section, we introduce the notions of a filter and a prime filter of the positive part $P = \{x \in A \mid x \geq 0\}$ of A.

Definition 4. A nonvoid subset F of P is called a *filter* of P if and only if

(i) If $x \in F$ and $y \in F$, then $x \cap y \in F$, and

(ii) If $y \in F$ and $y \leq x$, then $x \in F$

Definition 5. A proper filter Q in P is called a *prime filter* in P if and only if $x \in P$, $y \in P$ and $x + y \in Q$ imply $x \in Q$ or $y \in Q$.

It is easy to observe that the notions of a filter and a prime filter in P of A reduce to those defined by Rama Rao [2] for a DRℓ-semigroup $A = (A, +, \leq, -)$ satisfying $a - b \cup c = (a - b) \cap (a - c)$ for all a, b, c in A and to those defined by Banaschewski [1] for an ℓ-group, when A turns out to be a commutative DRℓ-semigroup and a commutative ℓ-group, respectively.

We recall the following

Definition 6 ([6]). Let $S = (S, +, \leq, *)$ and $T = (T, +, \leq, *)$ be two lattice ordered autometrized algebras. We say that a mapping $f : S \to T$ is a *homomorphism* if and only if f satisfies

(H1) $f(a + b) = f(a) + f(b)$ for all a, b in S,

(H2) $f(a * b) = f(a) * f(b)$ for all a, b in S,

(H3) $f(a \cup b) = f(a) \cup f(b)$ for all a, b in S, and

(H4) $f(a \cap b) = f(a) \cap f(b)$ for all a, b in S.

Further, a homomorphism $f : S \to T$ is called a *monomorphism* if and only if f is a 1 - 1 mapping and is called an eqimorphism if and only if f is onto.

The prime filters in P play a key role in our study, as we shall see in the following

Theorem 1. *For any nonzero homomorphism f of A into a totally ordered representable autometrized algebra T, $Q(f) = \{x \in P \mid f(x) > 0\}$ is a prime filter in P and* $\ker f = \{x \in A \mid x * 0 \notin Q(f)\}$. *Conversely, for any prime filter Q in P, there exists an epimorphism f from A onto a totally ordered representable autometrized algebra such that $Q = Q(f)$, namely the natural homomorphism $A \to A/K$, where K is the ideal $\{x \in A \mid x * 0 \notin Q\}$.*

Proof. Since f is a nonzero homomorphism, $Q(f)$ is nonempty. If $x \in Q(f)$ and $y \in Q(f)$, then $x \cap y \in Q(f)$ since $f(x \cap y) = f(x) \cap f(y) = f(x)$ or $f(y)$ (as T is totally ordered) and $f(x) > 0$ and $f(y) > 0$. Also, if $y \in Q(f)$ and $y \leq x$, then, since $f(x) \geq f(y) > 0$, it follows that $x \in Q(f)$. Hence $Q(f)$ is a filter in P. Now, let $x \in P$, $y \in P$ and

$x + y \in Q(f)$. Then, $f(x) \geq 0$, $f(y) \geq 0$ and $f(x) + f(y) = f(x + y) > 0$. Hence, either $f(x) > 0$ or $f(y) > 0$, so that $x \in Q(f)$ or $y \in Q(f)$. This implies that $Q(f)$ is a prime filter of P (observe that $Q(f)$ is a proper filter since $0 \notin Q(f)$).

If $x \in \ker f$, then, $f(x * 0) = f(x) * f(0) = 0 * 0 = 0$, and so $x * 0 \notin Q(f)$. Also, if $x * 0 \notin Q(f)$, then $f(x * 0) = 0$, which implies $f(x) * f(0) = 0$ from which follows that $f(x) = f(0) = 0$ so that $x \in \ker f$. Hence, we have $\ker f = \{x \in A \mid x * 0 \notin Q(f)\}$.

Conversely, let Q be a prime filter in P and let $K = \{x \in A \mid x * 0 \notin Q\}$. If $x \in K$ and $y \in K$, then, since $x * 0 \notin Q$ and $y * 0 \notin Q$, and Q is a prime filter, it follows that $x * 0 + y * 0 \notin Q$ which implies $x + y \in K$ (for, if $x + y \notin K$, then, since $(x + y) * 0 \in Q$, $(x + y) * 0 \leq x * 0 + y * 0$ (by (1.2)) and Q is a filter, we get $x * 0 + y * 0 \in Q$). Also, if $x \in K$, $y \in A$ and $y * 0 \leq x * 0$, then, since Q is a filter and $x * 0 \notin Q$, we get $y * 0 \notin Q$ so that $y \in K$. This shows that K is an ideal of A. Now, define a relation $\equiv$ on A by $a \equiv b(K)$ if and only if $a * b \in K$. Then, $\equiv$ is a congruence relation on A and the quotient A/K (the set of all equivalence classes relative to $\equiv$) is a representable autometrized algebra satisfying (1.3). Let $f : A \to A/K$ be the natural mapping $x \mapsto \overline{x}$ (where $\overline{x}$ is the equivalence class containing x). Then, clearly, f is an epimorphism. Now we show that $Q = Q(f)$. If $x \in Q$, then, since $x = x * 0$, $f(x) = \overline{x} \geq \overline{0}$ and $x * 0 \notin K$, it follows that $\overline{x} \neq \overline{0}$ so that $f(x) = \overline{x} > \overline{0}$ which implies $x \in Q(f)$. Also, if $x \in Q(f)$, then, since $x \in P$ and $\overline{x} > \overline{0}$, it follows that $\overline{x} \neq \overline{0}$ which implies $x \notin K$ and hence $x \in Q$. Thus $Q = Q(f)$.

It remains to be shown that A/K is totally ordered. Suppose A/K is

not totally ordered. Then, there exist $\overline{a}$ and $\overline{b}$ in A/K such that $(\overline{a} \cup \overline{b}) * \overline{a} \neq \overline{0}$ and $(\overline{a} \cup \overline{b}) * \overline{b} \neq \overline{0}$. This implies $(a \cup b) * a \not\equiv 0(K)$ and $(a \cup b) * b \not\equiv 0(K)$, which, by semiregularity, yields $(a \cup b) * a \notin Q$ and $(a \cup b) * b \notin K$. This implies $(a \cup b) * a \in Q$ and $(a \cup b) * b \in Q$ (by semiregularity of $*$). Since Q is a filter, we get $0 = ((a \cup b) * a) \cap ((a \cup b) * b) \in Q$ (by (1.3)), which shows that Q is not a proper filter, contradicting the fact that Q is a prime filter. Hence, A/K is a totally ordered representable autometrized algebra satisfying (1.3).

Remark 1. The above theorem is similar to a theorem (Proposition 1, [1]) proved by Banaschewski for a lattice ordered group and to a theorem proved by Rama Rao [2] for a DRℓ-semigroup satisfying $a - b \cup c = (a - b) \cap (a - c)$. Now, we get the theorems of Banaschewski and Rama Rao for commutative cases, as corollaries from our theorem.

Corollary 1 (Banaschewski). *For any nonzero homomorphism f of a commutative ℓ-group G into a totally ordered commutative ℓ-group T, $Q(f) = \{x \in P \mid f(x) > 0\}$ is a normal prime filter and* $\ker f = \{x \in G \mid |x| \notin Q(f)\}$. *Conversely, for any normal prime filter Q in P, there exists an epimorphism f from G onto a totally ordered commutative ℓ-group such that $Q = Q(f)$, namely the natural homomorphism $G \to G/K$ where K is the ℓ-ideal $\{x \in G \mid |x| \notin Q\}$.*

Corollary 2 (Rama Rao). *For any nonzero homomorphism f of a commutative DRℓ-group $A = (A, +, \leq, -)$, in which $a - b \cup c = (a - b) \cap (a - c)$ holds, into a totally ordered such DRℓ-group T, $Q(f) = \{x \in P \mid f(x) > 0\}$ is a normal prime filter in P and* $\ker f = \{x \in A \mid |x| \notin Q(f)\}$, *where* $|x|$

$= x * 0$. *Conversely, for any normal prime filter* Q *in* P *there exists an epimorphism* f *from* A *onto a totally ordered DRℓ-group such that* $Q = Q(f)$, *namely the natural map* $A \to A/K$, *where* K *is the ideal* $\{x \in A \mid |x| \in Q\}$.

Now, we introduce the following

Definition 7. An epimorphism $f : A \to T$, where T is a totally ordered representable autometrized algebra satisfying (1.3), is called *composite* if and only if there exist epimorphisms $f' : A \to T'$ and $g : T' \to T$, where T' is a totally ordered representable autometrized algebra, such that $f = g \circ f'$ and $\ker g \neq \{0\}$, and *irreducible* otherwise.

The relation between epimorphisms $f : A \to T$ and filters in P stated in Theorem 1 permits some more further remarks about such epimorphisms. In fact, we have the following characterization of irreducibility.

Corollary 3. *An epimorphism* $f : A \to T$, *where* T *is a totally ordered representable autometrized algebra, is irreducible if and only if* $Q(f)$ *is a maximal prime filter in* P.

Proof. Let $f : A \to T$ be composite. Then, since f is an epimorphism, it follows from Theorem 1, that $Q(f)$ is a prime filter in P. Since f is composite, there exists a totally ordered representable autometrized algebra T' and epimorphisms $f' : A \to T'$ and $g : T' \to T$ such that $f = g \circ f'$ and $\ker g \neq \{0\}$. This implies that $Q(f')$ is a prime filter in P and $\ker f' \subset \ker f$ (properly), which, in turn, implies $Q(f) \subset Q(f')$ (properly). This shows that $Q(f)$ is not a maximal prime filter in P.

Conversely, assume that the prime filter $Q(f)$ is not maximal. So, there exists a prime filter Q of P such that $Q(f) \subset Q$ (properly). Now, let $K = \{x \in A \mid x * 0 \notin Q\}$. Then, clearly, K is an ideal of A and $K \subset \ker f$ (properly) and the quotient $T' = A/K$ is a totally ordered representable autometrized algebra satisfying (1.3). Define $g : T' \to T$ by $g(\overline{a}) = f(a)$. g is a well defined mapping, for, if $\overline{a} = \overline{b}$, then, since $a * b \in K$, $a * b \in \ker f$, so that $f(a) = f(b)$. Clearly g is an epimorphism. Since $K \subset \ker f$, there exists $x \in P$ such that $x \in \ker f$ and $x \notin K$. This implies $g(\overline{x}) = f(x) = 0$ and $\overline{x} \neq \overline{0}$ so that $\ker g \neq \{0\}$. Let $f' : A \to T'$ be the natural mapping defined by $f'(a) = \overline{a}$. Then, f' is clearly, an epimorphism and $g \circ f' = f$, since $(g \circ f')(a) = g(f'(a)) = g(\overline{a}) = f(a)$ for each $a \in A$. Thus, f is composite.

Another consequence of Theorem 1, is the following factorization statement.

Corollary 4. *For any epimorphism $f : A \to T$, where T is a totally ordered representable autometrized algebra, there exists a totally ordered representable autometrized algebra T_0 and an irreducible epimorphism $f_0 : A \to T_0$ and an epimorphism $f_1 : T_0 \to T$ such that $f = f_1 \circ f_0$.*

Proof. Let $f : A \to T$ be the given epimorphism. Then, $Q(f) = \{x \in P \mid f(x) > 0\}$ is a prime filter in P. Let $\mathcal{F}$ be the set of all prime filters which contain $Q(f)$ and let it be partially ordered by set inclusion. If $\mathcal{C}$ is any chain in $\mathcal{F}$ and $F = \bigcup_{Q \in \mathcal{C}} Q$, then, by simple verification, we observe that $F \in \mathcal{F}$. Hence by Zorn's lemma, there exists a maximal prime filter M containing $Q(f)$. Now, let $K = \{x \in A \mid x * 0 \notin M\}$.

Clearly, K is an ideal of A and $K \subset \ker f$ (properly) (since $Q(f) \subset M$). Put $T_0 = A/K$, which is a totally ordered representable autometrized algebra satisfying (1.3). Define $f_0 : A \to T_0$ by $f_0(a) = \overline{a}$ and $f_1 : T_0 \to T$ by $f_1(\overline{a}) = f(a)$. It is easy to observe that f_1 is well defined (as in Corollary 3) and f_0 and f_1 are epimorphisms. We, now, show that $M = Q(f_0)$. If $x \in Q(f_0)$, then since $\overline{x} > \overline{0}$, we see that $x \not\equiv 0(K)$ which implies $x = x * 0 \notin K$ so that $x \in M$. Also, if $x \in M$, then, since $x = x * 0 \notin K$, we have $x \not\equiv 0(K)$ which implies $f_0(x) = \overline{x} > \overline{0}$ so that $x \in Q(f_0)$. Hence $M = Q(f_0)$. Thus $Q(f_0)$ is a maximal prime filter in P and hence f_0 is an irreducible epimorphism (by Corollary 3). Further, $f = f_1 \circ f_0$ since $f_1 \circ f_0(a) = f_1(f_0(a)) = f_1(\overline{a}) = f(a)$ for each $a \in A$.

The following results of Banaschewski and Rama Rao will follow from the above corollaries.

Corollary 5 (Banaschewski). *The following statements hold:*

(i) *An epimorphism $f : G \to T$, where G is a commutative ℓ-group and T is a totally ordered commutative ℓ-group, is irreducible if and only if $Q(f)$ is a maximal normal prime filter in P.*

(ii) *For any homomorphism $f : G \to T$, where G is a commutative ℓ-group and T is a totally ordered commutative ℓ-group, there exist a totally ordered commutative ℓ-group T_0, an irreducible epimorphism $f_0 : G \to T_0$ and an epimorphism $f_1 : T_0 \to T$ such that $f = f_1 \circ f_0$.*

Corollary 6 (Rama Rao). *The following statements hold:*

(i) *An epimorphism $f : A \to T$, where A and T are commutative DRℓ-semigroups satisfying $a - b \cup c = (a - b) \cap (a - c)$ and T is totally or-*

dered, is irreducible if and only if $Q(f)$ *is a maximal normal prime filter in* P.

(ii) *For any homomorphism* $f : A \to T$, *where* A *and* T *are commutative DRℓ-semigroups which satisfy* $a - b \cup c = (a - b) \cap (a - c)$ *and* T *is totally ordered, there exist a totally ordered commutative DRℓ-semigroup* T_0, *an irreducible epimorphism* $f_0 : A \to T_0$ *and an epimorphism* $f_1 : T_0 \to T$ *such that* $f = f_1 \circ f_0$.

§ 2. In this section, we are concerned with realizations of a representable autometrized algebra $A = (A, +, \leq, *)$ satisfying (1.3).

Following Rama Rao we introduce the following:

Definition 8. A monomorphism f from A into the direct product $\prod_{\alpha \in I} A_\alpha$ of nontrivial totally ordered representable autometrized algebras A_α, $(\alpha \in I)$ each of which satisfies (1.3), such that the image $f(A)$ is a subdirect product (i.e. the projections are onto) is called a *realization*.

Let $A = (A, +, \leq, *)$ be a representable autometrized algebra satisfying (1.3) and assume that A is a subdirect product of nontrivial totally ordered representable autometrized algebras T_α, $\alpha \in I$, each of which satisfies (1.3). That is, A is a subalgebra of the direct product $\prod_{\alpha \in I} T_\alpha$ such that $p_\alpha(A) = T_\alpha$ for all $\alpha \in I$, where $p_\alpha : \prod_{\alpha \in I} T_\alpha \to T_\alpha$ is the α-th projection. For each $u \in A$, let $S(u) = \{\alpha \in I \mid u(\alpha) \neq 0\}$. Since each T_α is nontrivial and the projections are onto, if $\beta \in I$, then there exists a nonzero $x \in T_\beta$ and $u \in A$ such that $u(\beta) = x \neq 0$. Hence $I = \bigcup_{u \in A} S(u)$. Now, if $\alpha \in S(u) \cap S(v)$, then, $u(\alpha) \neq 0$ and $v(\alpha) \neq 0$ so that $(u \cap v)(\alpha) = u(\alpha) \cap v(\alpha) \neq 0$ (since T_α is totally ordered) which implies $\alpha \in S(u \cap v)$.

On the other hand, if $\alpha \in S(u \cap v)$, then, since $(u \cap v)(\alpha) \neq 0$, $u(\alpha) \cap v(\alpha) \neq 0$ so that $u(\alpha) \neq 0$ and $v(\alpha) \neq 0$, since u and v may be assumed to be positive because $S(u) = S(u * 0)$ ($\alpha \in S(u)$ implies $(u * 0)(\alpha) \neq 0$ which implies $\alpha \in S(u * 0)$; and $\alpha \in S(u * 0)$ implies $u(\alpha) * 0(\alpha) \neq 0$ so that $u(\alpha) \neq 0$ and so $\alpha \in S(u)$). Thus, $\alpha \in S(u \cap v)$ implies $\alpha \in S(u) \cap S(v)$. Thus, $S(u \cap v) = S(u) \cap S(v)$ for all $u, v \in A$. Now, let $u, v \geq 0$ in A. If $\alpha \in S(u \cup v)$, then $u(\alpha) \cup v(\alpha) = (u \cup v)(\alpha) > 0$, which implies $u(\alpha) > 0$ or $v(\alpha) > 0$ (since T_α is totally ordered) so that $\alpha \in S(u) \cup S(v)$. And if $\beta \in S(u) \cup S(v)$, then $u(\beta) > 0$ or $v(\beta) > 0$ which implies that $(u \cup v)(\beta) = u(\beta) \cup v(\beta) > 0$ (since T_β is totally ordered) so that $\beta \in S(u \cup v)$. Hence $S(u \cup v) = S(u) \cup S(v)$ for any $u, v \geq 0$ in A. Thus the sets $S(u)$, $u \in A$ will form a base for a topology $\mathcal{D}$ on I. A will be called *zero-dimensional* if and only if the $S(u)$, $u \in A$, are all closed (besides being open by definition) in this topology $\mathcal{D}$ on I. We obtain a criterion for A to have this property in the following

Theorem 2. *A is zero-dimensional if and only if for each $\alpha \in I$, the prime filter $Q_\alpha = \{u \in P \mid u(\alpha) > 0\}$ in P, is an ultrafilter.*

Proof. Let A be zero-dimensional. Using the fact that each T_α is totally ordered, by direct verification, we observe that for each $\alpha \in I$, Q_α is a prime filter in P. Let $\alpha \in I$. We show that Q_α is an ultrafilter. Since Q_α is a prime filter, there exists $u \in P$ such that $u \notin Q$. This implies $u(\alpha) = 0$ so that $\alpha \notin S(u)$. Since each $S(u)$ is closed by hypothesis, there exists $S(v)$, $v \in A$ such that $\alpha \in S(v)$ and $S(u) \cap S(v) = \emptyset$. We may assume $v \geq 0$ in A (since $S(v) = S(v * 0)$) so that $v(\alpha) >$

0. This implies that there exists $v \in Q_\alpha$ such that $u \cap v = 0$.

Suppose $Q_\alpha \subset B$ (properly), where B is a filter in P. Then there exists $x \in B$ such that $x \notin Q_\alpha$. Hence, as has been done above, there exists $y \in Q_\alpha$ such that $x \cap y = 0$. Since $Q_\alpha \subset B$ and $y \subset Q_\alpha$, we have $y \in B$. This shows that $0 = x \cap y \in B$ (since B is a filter) so that $B = P$. Hence Q_α is an ultrafilter.

Conversely, assume that for each $\alpha \in I$, Q_α is an ultrafilter. We show that for each $u \in A$, $S(u)$ is closed (in I). Let $\alpha \in I \setminus S(u)$. Then, $\alpha \notin S(u)$ so that $u(\alpha) = 0$. Since $u \geq 0$ may be assumed (since $S(u) = S(u * 0)$), $u \notin Q_\alpha$. Since Q_α is an ultrafilter and $u \notin Q_\alpha$, there exists $v \in Q_\alpha$ such that $u \cap v = 0$ or $S(u) \cap S(v) = \emptyset$. This implies that α is not a limit point of $S(u)$. That is, $S(u)$ contains all of its limit points. Hence $S(u)$ is closed, for each $u \in A$. Hence A is zero-dimensional.

Now, we have the following theorems of Banaschewski and Rama Rao for commutative cases, as corollaries.

Corollary 7 (Banaschewski). *If G is a commutative ℓ-group such that G is a subdirect product of totally ordered commutative ℓ-groups T_α, $\alpha \in I$, then G is zero-dimensional if and only if each $Q_\alpha = \{u \in P \mid u(\alpha) > 0\}$ is an ultrafilter.*

Corollary 8 (Rama Rao). *If A is a commutative DRℓ-semigroup in which $a - b \cup c = (a - b) \cap (a - c)$ holds, such that A is a subdirect product of totally ordered such DRℓ-semigroups T_α, $\alpha \in I$, then, A is zero-dimensional if and only if for each $\alpha \in I$, the normal prime filter Q_α =*

$\{u \in P \mid u(\alpha) > 0\}$ *in* P *is an ultrafilter.*

The following example due to Banaschewski [1] shows that a subdirect product of nontrivial totally ordered representable autometrized algebras T_α, $\alpha \in I$ need not be zero-dimensional.

Example. Let G be the additive group of all continuous real functions on a completely regular connected Hausdorff space E, partially ordered as usual. Then G is a subdirect product of totally ordered groups T_x, $x \in E$, where for each $x \in E$, $T_x = \mathbb{R}$, the additive group of reals in its natural order. For each $u \in G$, $S(u)$ is open and the topology of E is generated by these sets; hence no $S(u)$ distinct from $\emptyset$ and E is also closed, since E is connected. Thus G is not zero-dimensional. Now, choose $E = \mathbb{R}$, then, none of the normal prime filters Q_x, $x \in E$, is an ultrafilter in P (for details, see [1]).

Definition 9. Let $f : A \to \prod_{\alpha \in I} A_\alpha$ be a realization of A. Then, we say that f is a *zero-dimensional realization* if and only if $f(A)$ is zero-dimensional.

The following theorem establishes the existence of a zero-dimensional realization for any representable autometrized algebra $A = (A, +, \leq, *)$ satisfying (1.3).

Theorem 3. *A has a zero-dimensional realization.*

We need the following

Lemma 1. *Each ultrafilter in P is prime.*

Proof. Let U be an ultrafilter in P and $x, y \in P$ such that $x \notin U$ and $y \notin U$. Then, there exists $a, b \in U$ such that $x \cap a = 0$ and $y \cap b = 0$. This implies $c = a \cap b \in U$ and $(x + y) \cap c = 0$ (since $x \cap c = 0 = y \cap c$ and $A = (A, +, \leq)$ is a lattice ordered semigroup). Hence $x + y \notin U$ (for, if $x + y \in U$, then, since $c \in U$, $0 = (x + y) \cap c \in U$, contradicting the fact that U is an ultrafilter). Hence U is a prime filter.

Proof of Theorem 3. Let $\Omega = \{U \mid U$ is an ultrafilter in $P\}$. If a is a nonzero element in P, then, a belongs to the filter generated by a and hence to some ultrafilter U, by Zorn's lemma. Thus $\Omega \neq \emptyset$. For each $U \in \Omega$, consider the ideal $K(U) = \{x \in A \mid x * 0 \notin U\}$ and the natural homomorphism $h_U : A \to T(U) = A/K(U)$. Then, $\bigcap_{U \in \Omega} K(U) = \{0\}$, since each nonzero a in P belongs to some ultrafilter. Now, define $h : A \to \prod_{U \in \Omega} T(U)$ by $h(x) = (h_U(x))_{U \in \Omega}$. If $h(x) = h(y)$, since $\overline{x}_U = \overline{y}_U$ for every $U \in \Omega$, we have $x_U * y_U \in K(U)$ for every $U \in \Omega$ so that $x_U = y_U$ for every $U \in \Omega$ (since $\bigcap_{U \in \Omega} K(U) = \{0\}$) which implies $x = y$. This shows that h is a 1 - 1 mapping. Further, $h(x + y) = (h_U(x + y))_{U \in \Omega} = (\overline{x + y}_U)_{U \in \Omega} = (\overline{x}_U + \overline{y}_U)_{U \in \Omega} = (\overline{x}_U)_{U \in \Omega} + (\overline{y}_U)_{U \in \Omega} = (h_U(x))_{U \in \Omega} + (h_U(y))_{U \in \Omega} = h(x) + h(y)$. Similarly, we can show that $h(x \cup y) = h(x) \cup h(y)$, $h(x \cap y) = h(x) \cap h(y)$ and $h(x * y) = h(x) * h(y)$ for all $x, y \in A$. Hence h is a monomorphism. Now, if p_U denotes the projection from $\prod_{U \in \Omega} T(U)$ to $T(U)$, then, for each $x \in A$, we have $(p_U \circ h)(x) = p_U((h_U(x))_{U \in \Omega}) = h_U(x) = \overline{x}_U$. This shows that $p_U \circ h$ is onto, for each $U \in \Omega$. Hence h is a realization. Next, for each $U \in \Omega$, we observe that $U = \{x \in P \mid h_U(x) > \overline{0}_U\}$. For, if $x \in P$ and $h_U(x) > \overline{0}_U$, since $\overline{x}_U > \overline{0}_U$, we get $x \notin K(U)$ so that $x \in U$. And, if

$x \in U$, since $x \in P$ and $x = x * 0 \notin K(U)$, we have $\overline{x}_U > \overline{0}_U$ so that $h_U(x) > \overline{0}_U$. Thus, for each $U \in \Omega$, $\{x \in P \mid h_U(x) > \overline{0}_U\}$ is an ultrafilter in P. So, from Theorem 2, follows that $h(A)$ is zero-dimensional. So, by Definition 9, follows that h is a zero-dimensional realization.

Corollary 9. *If* $A = (A, +, \leq, -)$ *is a commutative DRℓ-semigroup satisfying* $a - b \cup c = (a - b) \cap (a - c)$ *for all* a, b, c *in* A, *then,* A *has a zero-dimensional realization.*

Corollary 10 (Ribenboim). *Every commutative ℓ-group has a zero-dimensional realization.*

§ 3. If $A = (A, +, \leq, *)$ is a representable autometrized algebra satisfying the conditions

(1.3) $(a * (a \cup b) \cap (b * (a \cup b)) = 0$ for all a, b in A, and

(1.4) $(a * (a \cap b)) + a \cap b = a$ for all a, b in A,

then, we have proved elsewhere [5] that $(A, \leq)$ is a distributive lattice and consequently we get that P is a distributive lattice with zero and the theorem of Banaschewski [1] for commutative ℓ-groups and that of Rama Rao [2] for commutative DRℓ-semigroups regarding the lattice of filets can be straightaway extended to a representable autometrized algebra $A = (A, +, \leq, *)$ satisfying (1.3) and (1.4). In fact, we get

Theorem 4. *For any realization* $f : A \to \prod_{\alpha \in I} T_\alpha$, *where each* T_α *is a totally ordered representable autometrized algebra satisfying* (1.3) *and* (1.4), *the correspondence* $\overline{x} \mapsto \Gamma S(x)$ *is a one-to-one join preserving mapping. In particular, if the realization is zero-dimensional, then, this*

correspondence reduce to $\overline{x} \mapsto S(x)$ *and is a lattice monomorphism.* (Γ *denotes the closure operator in the topology on* I, *as in* [1]).

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