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Complemented Semilattices

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A meet semilattice L with least element 0 and greatest element 1 is said to be complemented if for any x in L there exists an element y (complement of x) in L such that the join $x \vee y$ of x and y exists in L and is equal to 1 and the meet $x \wedge y$ of x and y in L is zero. L is called uniquely complemented if each element in L has a unique complement. In [5] J. Varlet has proved that a bounded distributive semilattice L is a Boolean algebra if and only if L is complemented. In this note we observe that every complemented very weakly distributive semilattice is a Boolean algebra. Further we give a set of equivalent conditions for a uniquely complemented semilattice to be a Boolean algebra. Finally we prove that a dual semicomplemented semilattice L with 0 and 1 in which every element of L is the join of finite number of join primes, is a finite Boolean algebra.

Throughout this paper we shall be concerned with meet semilattices. We follow the definitions given in [5] and [6]. A semilattice L is *distrib-*

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utive if $z \geq x \wedge y$ ($x, y, z \in L$) implies the existence of x_1, y_1 in L such that $x_1 \geq x, y_1 \geq y$ and $x_1 \wedge y_1 = z$. A semilattice L is *modular* if, for any x, y, z in L such that $y \geq z \geq x \wedge y$, there exists $x_1 \geq x$ satisfying $z = x_1 \wedge y$. A semilattice S is *weakly distributive* if $\forall \{x_i : i \in I, I \text{ is finite}\}$ exists in L then for each $x \in L, \bigvee \{x \wedge x_i : i \in I\}$ exists and equals $x \wedge (\bigvee \{x_i : i \in I\})$. A semilattice L is *very weakly distributive*, if $x_1 \vee x_2$ exists in L , then for all x of $L, (x \wedge x_1) \vee (x \wedge x_2)$ exists and equals $x \wedge (x_1 \vee x_2)$. A semilattice L with 0 is *weakly complemented* if for any pair a, b ($a < b$) of elements of L , there is an element c such that $a \wedge c = 0$ and $b \wedge c \neq 0$. A semilattice L with 0 is called *weakly 0-distributive* (in the sense of Y. S. Pawar and N. K. Thakare [4] who uses the expression 0-distributive semilattices) if for any $a, b, c \in L, a \wedge b = 0 = a \wedge c$ imply $a \wedge (b \vee c) = 0$ whenever $b \vee c$ exists in L . We notice that this concept is weaker than that used in [5] under the same name. A semilattice L with 0 and 1 is said to be *dual semicomplemented* if for each $0 \neq a \in L$, there is $b \in L$ such that $b \neq 1, a \vee b$ exists and is equal to 1 . It can be easily seen that every complemented semilattice is dual semicomplemented. A non zero element $a \in L$ is an atom if for any $x \in L, 0 \leq x \leq a$ implies either $0 = x$ or $x = a$.

§ 1 Complemented semilattices. For the rest of this paper L will denote a meet semilattice with 0 and 1 . If any pair $x, y \in L$ has a least upper bound, this will be denoted by $x \vee y$. For any $x \in L, x'$ denotes the unique complement (if exists) of x in L .

Proposition 1. *Let L be a weakly complemented semilattice. Then the*

following statements are equivalent:

- (i) L is very weakly distributive.
- (ii) For each $0 \neq x \in L$ there exists $0 \neq d \leq x \in L$ such that for every $a, b \in L$, whenever $a \vee b$ exists, $(d \wedge a) \vee (d \wedge b)$ exists and is equal to $d \wedge (a \vee b)$.
- (iii) L is weakly 0-distributive.

Proof. (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii): Suppose (ii) holds. If L is non weakly 0-distributive, then there exists $a, b, c \in L$ such that $a \wedge b = 0 = a \wedge c$, $b \vee c$ exists and $a \wedge (b \vee c) \neq 0$. Now by (ii) there is $0 \neq d \in L$ such that $d \leq a \wedge (b \vee c)$, $(d \wedge b) \vee (d \wedge c)$ exists and equals $d \wedge (b \vee c)$. Since $d \leq a$, it follows that $d \wedge b = 0 = d \wedge c$. Consequently $d \wedge (b \vee c) = 0$. As $d \leq b \vee c$ we have $d = 0$ which is impossible. Hence L is weakly 0-distributive.

(iii) $\Rightarrow$ (i): Suppose (iii) holds. Let $x, y, z \in L$ and $y \vee z$ exists. Clearly $x \wedge (y \vee z)$ is an upper bound of $(x \wedge y)$ and $(x \wedge z)$. Let $(x \wedge y), (x \wedge z) \leq w$ and $x \wedge (y \vee z) \neq w$ for some $w \in L$. Then $(x \wedge (y \vee z)) \wedge w < x \wedge (y \vee z)$ and so by weakly complementation, there is $t \in L$ such that $x \wedge (y \vee z) \wedge w \wedge t = 0$ and $x \wedge (y \vee z) \wedge t \neq 0$. As $w \geq x \wedge y, x \wedge z$ it follows that $x \wedge y \wedge t = 0 = x \wedge z \wedge t$. Since $y \vee z$ exists, by (iii) we get $(x \wedge t) \wedge (y \vee z) = 0$. But this contradicts the fact that $(x \wedge t) \wedge (y \vee z) \neq 0$. Therefore $x \wedge (y \vee z)$ is the least upper bound of $x \wedge y$ and $x \wedge z$ and so (i) holds. This completes the proof of the proposition.

Proposition 2. *If L is a complemented very weakly distributive semilattice, then L is a Boolean algebra.*

Proof. By using the definition of a very weakly distributive semilat-

tice, one can easily shown that L is uniquely complemented and for every $a, b \in L$, $a \vee b$ exists in L and equals $(a' \wedge b')'$ and so L is a lattice. Consequently L is a complemented distributive lattice.

Corollary 1. *Every weakly 0-distributive complemented semilattice which is also weakly complemented is a Boolean algebra.*

Proof. This follows from Propositions 1 and 2.

Corollary 2. *Every uniquely complemented semilattice which is also weakly 0-distributive is a Boolean algebra.*

Proof. This follows from the fact that every uniquely complemented semilattice is a weakly complemented semilattice.

Corollary 3. *Every complemented modular semilattice L which is also weakly 0-distributive is a Boolean algebra.*

Proof. By Corollary 1, it is enough if we show that L is weakly complemented. Suppose $a < b$ for some $a, b \in L$. Let c be a complement of a in L . Clearly $a \wedge c = 0$. If $b \wedge c = 0$, then $b \geq a \geq b \wedge c = 0$, and so by modularity, $a = b \wedge d$ for some $d \geq c$. Since $d \geq a, c$, it follows that $d = 1$ and so $a = b$ which is absurd. Therefore $b \wedge c \neq 0$. Hence L is weakly complemented.

For each $a \in L$, we denote $[0, a] = \{x \in L \mid 0 \leq x \leq a\}$. L is called *sectionally complemented*, if each interval $[0, a]$ is complemented as a semilattice. It can be easily observed that every sectionally complemented semilattice is weakly complemented.

§ 2. Characterizations of Boolean algebras. In [3], C. C. Chen has given a set of conditions for a uniquely complemented lattice to be a Boolean algebra. The following Proposition 3 generalizes his result.

Proposition 3. Let L be a uniquely complemented semilattices. Then the following statements are equivalent:

- (i) *L is a Boolean algebra.*
- (ii) *L is a modular semilattice.*
- (iii) *If $c \leq b$, $c \vee a$ exists and is equal to 1 in L , then $c \vee (a \wedge b)$ exists and is equal to b in L .*
- (iv) *L is sectionally complemented.*
- (v) *For every $x, y \in L$, $x \leq y$ implies $y' \leq x'$.*
- (vi) *For each $a \in L$, $a \leq (a' \wedge b')$ for all $b \in L$.*
- (vii) *For every $a, b \in L$, $a \vee b$ exists and is equal to $(a' \wedge b')'$.*
- (viii) *For every $x, y \in L$, if $x \vee y$ exists and is equal to 1, then $x' \leq y$.*
- (ix) *For $x, y, z \in L$, if $x \vee y, x \vee z$ exist and equals 1, then $x \vee (y \wedge z)$ exists and is equal to 1.*

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii): Suppose $c \leq b$, $c \vee a$ exists and is equal to 1 for some $a, b, c \in L$. Clearly $b \geq c$, $a \wedge b$. Let $d \geq c$, $a \wedge b$. As $b \geq d \wedge b \geq b \wedge a$, by (ii), there is $e \in L$ such that $d \wedge b = b \wedge e$ and $e \geq a$. Since $e \geq a$ and $e \geq d \wedge b \geq c$, it follows that $e \geq c \vee a = 1$ and so $d \wedge b = b$. Therefore b is the least upper bound of c and $a \wedge b$. Thus (iii) holds.

(iii) $\Rightarrow$ (iv): Suppose (iii) holds. Let $a \in L$ and $b \in [0, a]$. Since $b \leq a$ and $b \vee b'$ exists and equals 1, by (iii), $b \vee (a \wedge b')$ exists and

is equal to a . Also $b \wedge (a \wedge b') = 0$. Hence $[0, a]$ is complemented.

(iv) $\Rightarrow$ (v): Suppose (iv) holds. Let x, y be any two elements of L such that $x \leq y$. Then $y \vee x'$ exists and is equal to 1. Consider the interval $[0, x']$. Let z be a complement of $x' \wedge y$ in $[0, x']$. Clearly $z \leq x'$. Now $z \wedge y = (z \wedge x') \wedge y = z \wedge (x' \wedge y) = 0$. Also if $t \geq z, y$, then $t \geq z, x' \wedge y$ and so $t \geq z \vee (x' \wedge y) = x'$. Since $t \geq x', y$, it follows that $t = 1$; so that $z \vee y$ exists and is equal to 1. Therefore z is a complement of y and so by uniqueness $z = y'$. Consequently $y' \leq x'$. Thus (v) holds.

(v) $\Rightarrow$ (vi) is obvious.

(vi) $\Rightarrow$ (vii): Suppose (vi) holds and let $a, b \in L$. Clearly by (vi) $a, b \leq (a' \wedge b')'$. Suppose $a, b \leq c$ for some $c \in L$. Then by (vi) $c' \leq (c \wedge a)' = a'$ and $c' \leq (c \wedge b)' = b'$; so that $c' \leq a' \wedge b'$. Again by (vi), $(a' \wedge b')' \leq ((a' \wedge b') \wedge c')' = c'' = c$. Therefore $(a' \wedge b')'$ is the least upper bound of a and b and thus (vii) holds.

(vii) $\Rightarrow$ (viii): Suppose (vii) holds. Let $x \vee y$ exists and be equal to 1 for some $x, y \in L$. Put $p = y \wedge (x \wedge y)'$. We claim that $(p \vee x) \wedge y$ is a complement of y' . Clearly $((p \vee x) \wedge y) \wedge y' = 0$. Now $((p \vee x) \wedge y) \vee y' \geq (x \wedge y) \vee y' = ((x \wedge y)' \wedge y)'$ (by vii) $= p'$ and $((p \vee x) \wedge y) \vee y' \geq p \wedge y = p$ and so $((p \vee x) \wedge y) \vee y' = 1$. Therefore $(p \vee x) \wedge y$ is a complement of y' and hence by uniqueness, $(p \vee x) \wedge y = y$. Since $x \vee y = 1$ and $y \leq p \vee x$, it follows that $p \vee x = 1$. Obviously $p \wedge x = 0$ and therefore $p = x'$. Consequently $x' \leq y$.

(viii) $\Rightarrow$ (ix) is obvious.

(ix) $\Rightarrow$ (i): Suppose (ix) holds. By Corollary 2, it is enough if we

show that L is weakly 0-distributive. Let $x \wedge y = 0 = x \wedge z$ and $y \vee z$ exists for $x, y, z \in L$. Let w be the complement of $x \wedge (y \vee z)$. Then $w \vee x$ and $w \vee (y \vee z)$ exist and are equal to 1. We claim that for any $1 \geq w, y, 1 \vee z$ exists and is equal to 1. If $m \geq 1, z$, then $m \geq w, y \vee z$ and so $m = 1$. Therefore $1 \vee z = 1$; also $1 \vee x = 1$ and hence by (ix) $1 \vee (x \wedge z)$ (exists) $= 1$. As $x \wedge z = 0$, it follows that $1 = 1$. This shows that 1 is the least upper bound of w and y . Again by (ix) $w \vee (y \wedge x)$ exists and is equal to 1. Since $x \wedge y = 0$, it follows that $w = 1$. Consequently $x \wedge (y \vee z) = 0$ and thus (1) holds. This completes the proof of the proposition.

In [1] V. K. Balachandran has proved that a complete, complemented lattices satisfying the condition: every element of L is the join of finite number of join primes, are precisely the finite Boolean algebras (see Theorems 2 and 3 in [1]). The following Proposition 3 is a generalization of his result.

Definition. An element a of a semilattice S is said to be *join prime* if $\bigvee \{a_i : i \in I, I \text{ is finite}\}$ exists and $a \leq \bigvee_{i \in I} a_i$ implies $a \leq a_i$ for some $i \in I$.

It can be easily seen that in lattices our definition of join prime will coincide with that given in [1].

Lemma 1. *Let L be dual semi complemented and a be a join prime element of L . Then a is an atom.*

Proof. Similar to (Lemma 2 [1]).

Proposition 3. *Let L be dual semi complemented. If every element of L is the join of finite number of join primes, then L is a finite Boolean algebra.*

Proof. Suppose L is dual semicomplemented and every element of L is the join of finite number of join primes. First we prove that L is weakly distributive. Let $\{x_i : i \in I, I \text{ is finite}\}$ be a set of finite number of elements of L such that $\bigvee\{x_i : i \in I\}$ exists in L . Clearly $x \wedge (\bigvee_{i \in I} x_i) \geq x \wedge x_i$ for every $i \in I$. Suppose $y \geq x \wedge x_i$ for every $i \in I$. By hypothesis, $x \wedge (\bigvee_{i \in I} x_i) = \bigvee_{j=1}^n a_j$, where a_j 's are join primes. Since for each a_j is join prime and $a_j \leq \bigvee\{x_i : i \in I\}$, it follows that $a_j \leq x_i$ for some $i \in I$; so that $a_j \leq x \wedge x_i$ for some $i \in I$ and hence $a_j \leq y$. Consequently $x \wedge (\bigvee_{i \in I} x_i) = \bigvee_{j=1}^n a_j \leq y$ and therefore $\bigvee_{i \in I} (x \wedge x_i)$ exists and is equal to $x \wedge (\bigvee_{i \in I} x_i)$. Thus L is weakly distributive. Also by Lemma 1, every element of L is the join of finite number of atoms. We claim that L is complemented. Let a be an atom in L . As L is dual semi-complemented and $a \neq 0$, there is $b \in L$ such that $b \neq 1$, $a \vee b$ exists and is equal to 1. Again since $a \not\leq b$, and a is an atom, we have $a \wedge b = 0$. Therefore b is a complement of a . By weakly distributivity, it can be easily shown that b is the unique complement of a . Let x be any arbitrary element of L and suppose $x = \bigvee_{i=1}^n a_i$ (a_i 's are atoms). Then $y = \bigwedge_{i=1}^n a_i'$ is the complement of x , where a_i' denotes the unique complement of a_i , for $i = 1, 2, \dots, n$; and so L is complemented. Further by Proposition 2, L is a Boolean algebra as L is complemented and weakly distributive. Now one can easily prove that every prime filter of L

is principal. So by dual of (Theorem 1 [2]: Let L be a distributive lattice with 1. If every prime ideal in L is principal, then every ideal is principal), every filter is principal. Consequently L is a finite Boolean algebra. This completes the proof of the proposition.

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