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A Generalization of α -Ideals

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Introduction. Unifying the ideal theories in various structures such as commutative semigroups, commutative rings, distributive lattices etc., K. E. Aubert [1] introduced the concept of α -ideals in a commutative semigroup and studied their properties. But, his theory does not cover structures like modules. In this paper we introduce the concepts of pre S -spaces and S -spaces, which also cover modules over commutative rings, continuous mappings of a topological space into itself, endomorphisms of a universal algebra and so on (see examples below), and generalize some of the results in [1]. In §1 we define pre S -spaces and S -spaces, and obtain some equivalent conditions of the continuity axiom. In §2 we consider an induced closure system in S when T is a pre S -space (S -space) and discuss their properties. Further results will be communicated later.

§ 1. Pre S -spaces. By a *closure operation* on a set T we mean a mapping $A \mapsto \bar{A}$ on the power set of T satisfying the following conditions:

- (i) $A \subset \bar{A}$.
- (ii) $A \subset \bar{B}$ implies $\bar{A} \subset \bar{B}$.

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For any subset A of T , $\bar{A}$ is called the *closure* of A and A is said to be *closed* if $\bar{A} = A$. The family of all closed sets in T is called the *closure system* corresponding to the closure operation. It is well-known that a necessary and sufficient condition for a family of subsets of T to be a closure system is that it is closed under arbitrary intersections.

Definition. Let S and T be sets. We say that T is equipped with *S-multiplication* if there is a map, denoted multiplicatively, from $S \times T$ into T .

Definition. Let S be a set. By a *pre S-space* is meant a set T together with an *S-multiplication* in T and a closure operation on T satisfying the "continuity axiom":

$$(P_1) \quad M\bar{A} \subset \overline{MA} \quad \text{for all } M \subset S \text{ and } A \subset T$$

or equivalently,

$$s\bar{A} \subset \overline{sA} \quad \text{for all } s \in S \text{ and } A \subset T.$$

Example 1. Let S be a family of continuous mappings on a topological space T . Define *S-multiplication* in T by $(h, t) \mapsto h(t)$, and take $\bar{A}$ to be the closure of A for each $A \subset T$.

Example 2. Let (U, Ω) be a universal algebra with a set M of operators, i.e. suppose that there exists a mapping $m \mapsto \bar{m}$ of M into the set of endomorphisms of (U, Ω) . Define *M-multiplication* in U by $(m, u) \mapsto \bar{m}(u)$, and for any $A \subset U$ let $\bar{A}$ be the sub-algebra generated by A . As particular cases we obtain several examples like (i) U is a group and M is a set of inner automorphisms (endomorphisms, automorphisms). (ii) U

is a left R -module, M is a subset of R , the usual left multiplication and for each $A \subset U$, $\bar{A}$ is the subgroup generated by A . (iii) U is a left R -module where R is a commutative ring, and for each $A \subset U$, $\bar{A}$ is the submodule generated by A . We can have similar examples with semilattices, lattices, ect.

Example 3. Suppose T is a join semilattice and S is a set of endomorphisms of T . Define S -multiplication in T by $(h, \alpha) \mapsto h(\alpha)$. Let $\bar{A}$ be the ideal generated by A for all $A \subset T$. As special cases we can take (i) a subset M of a join semilattice L and define $m\ell$ as $m \vee \ell$ for all $m \in M, \ell \in L$ and take $\bar{A}$ as the ideal generated by A (ii) a subset M of a distributive lattice L , define $m\ell$ as $m \wedge \ell$ for all $m \in M, \ell \in L$ and take $\bar{A}$ as in (i).

Example 4. Let S be a commutative semigroup. Then S is a pre S -space with the multiplication in S and the ideal generated by A for $\bar{A}$.

We have several other examples such as ideals in commutative rings etc., and in fact every example of an α -system as defined in Aubert [1] is an example for our pre S -spaces.

Definition. Let T be a pre S -space. $A \subset T$ is called an *ideal* if $\bar{A} = A$. The family of all ideals in T is called the ideal system of T .

We observe that the ideal system of T forms a complete lattice under set inclusion since it is closed under arbitrary intersections.

Notation. In what follows T denotes a set equipped with S -multiplication. M, N denote subsets of S and A, B, C denote subsets of T .

When $A \vdash \bar{A}$ is a closure operation on T , $M \circ A$ denotes $\overline{MA}$ and $A \vee B$ denotes $\overline{A \cup B}$ for all M, A, B . In [1], $\overline{A \cup B}$ is denoted by $A \cup_x B$.

The following lemma is a generalization of Theorem 1 of [1].

1.1. Lemma. *Let $A \vdash \bar{A}$ be a closure operation on T . Then the following are equivalent.*

- (i) *The continuity axiom: $\overline{MA} \subset \overline{M\bar{A}}$ for all M, A .*
- (ii) *$M \circ A = M \circ \bar{A}$ for all M, A .*
- (iii) *$M(A \vee B) \subset MA \vee MB$ for all M, A, B .*
- (iv) *$M \circ (A \vee B) = M \circ A \vee M \circ B$ for all M, A, B .*

This lemma may be proved by showing that in the sequence (i), (iii), (iv), (ii), (i), each implies the next. This is a routine check and hence is omitted.

Note that in (ii), (iii), (iv) of Lemma 1.1, M can be replaced by $\{m\}$.

1.2. Corollary. *Let T be a pre S -space. Then for any M the mapping $A \vdash M \circ A$ is a join semilattice endomorphism of the family of all ideals in T . As a particular case, if U is a group and M , a set of endomorphisms of U (cf. Example 2(i)) then (using the fact that the endomorphic image of a subgroup is a subgroup) each member of M induces a join semilattice endomorphism of the family of subgroups of U .*

Definition. The set $\{t \in T \mid Mt \subset A\}$ will be denoted by $A * M$. We write $A * m$ instead of $A * \{m\}$.

1.3. Remark. We observe that (i) $A \subset B$ implies $A * M \subset B * M$
(ii) $MB \subset A$ iff $B \subset A * M$ and in particular $M(A * M) \subset A$; also,

(iii) $(\bigcap_{i \in I} A_i) * M = \bigcap_{i \in I} (A_i * M)$ and (iv) $A * \bigcup_{i \in I} M_i = \bigcap_{i \in I} (A * M_i)$ where $A_i \subset T$ and $M_i \subset S$ for all $i \in I$.

The following lemma gives some other conditions equivalent to the continuity axiom.

1.4. Lemma. Let $A \vdash \bar{A}$ be a closure operation on T . Then the following are equivalent.

- (i) The continuity axiom
- (ii) $\overline{\bar{A} * M} = \bar{A} * M$ for all A, M
- (iii) $\overline{\bar{A} * m} = \bar{A} * m$ for all $A \subset T, m \in S$
- (iv) $\overline{\bar{A} * M} \subset \bar{A} * M$ for all A, M .

Proof. Assume (i), and put $B = \bar{A} * M$. Then $MB \subset \bar{A}$. Now by our assumption $\overline{MB} \subset \overline{MB} \subset \bar{A}$ and hence $\bar{B} \subset \bar{A} * M = B$ which gives (ii). The implication (ii) $\Rightarrow$ (iii) is obvious. Assume now (iii). Let F denote the family of all closed sets in T . By our assumption $\bar{A} * m \in F$ for all $m \in M$ and hence $\bar{A} * M$, which means $\bigcap_{m \in M} \bar{A} * M$ also belongs to F . Since $MA \subset \overline{MA}$, we have $A \subset \overline{MA} * M$ which belongs to F , and hence $\bar{A} \subset \overline{MA} * M$ which gives that $\overline{MA} \subset \bar{A}$, proving (i). Hence (i), (ii), (iii) are equivalent. Finally, assume (ii). Since $A \subset \bar{A}$, we have $A * M \subset \bar{A} * M$ which, using our assumption, gives (iv). It is easy to see that (iv) implies (ii). Hence the proof is completed.

1.5. Corollary (cf. Theorem 3 of [1]). Let T be a commutative semi-group. Let $A : B$ denote the set $\{c \in T \mid cB \subset A\}$ for all $A, B \subset T$. Let the map $A \vdash A_x$ be a closure operation on T . Then the following are equivalent.

(i) *The continuity axiom.*

(ii) $(A_x : B)_x = A_x : B$ for all $A, B \subset T$

(iii) $(A_x : b)_x = A_x : b$ for all $A \subset T, b \in T$

(iv) $(A : B)_x \subset A_x : B$ for all $A, B \subset T$.

Proof. Since T is commutative, $A : B$ is the same as $A * B$ when T is considered as a set with T -multiplication. Now the result is a direct consequence of Lemma 1.4.

1.6. Remark. Applying Lemma 1.1 and 1.4 to Example 1 we get the following well-known equivalent conditions for the continuity of a mapping h of a topological space T into itself: (a) $h(\overline{A}) \subset \overline{h(A)}$ for all $A \subset T$; (b) $h^{-1}(A)$ is closed whenever A is a closed subset of T ; (c) $\overline{h^{-1}(A)} \subset h^{-1}(\overline{A})$ for all $A \subset T$.

1.7. Remark. In the case of an Ω -algebra U (see Example 2) let $\overline{A}$ denote the subalgebra generated by A . If h is an endomorphism of U Lemma 1.4 says that $h(\overline{A}) \subset \overline{h(A)}$ for all $A \subset U$ iff $h^{-1}(B)$ is a subalgebra of U whenever so is B .

Similar interpretations are possible with regards to modules, lattices, etc.

Definition. Let T be a pre S -space. The ideal system in T is said to be *algebraic* (or of *finite character*) if for each $A \subset T$, $\overline{A}$ coincides with the union of all $\overline{F}$'s whenever F runs through finite subsets of A .

The following lemma, which may be proved by arguing as in Theorem 6 of [1], generalizes the latter.

1.8. Lemma. *The ideal system in a pre S -space is algebraic iff the union of any chain of ideals is an ideal.*

In fact Lemma 1.8 holds for any closure system in a set, as evidenced by the proof given in Aubert ([1]). For an alternate proof one may see Theorem 1.2, Chapter II of [2].

Definition. Let T be a pre S -space. We say that T is an S -space if

$$(P_2) \quad M\bar{A} \subset \bar{A} \text{ for all } M, A$$

which is equivalent to " $m\bar{A} \subset \bar{A}$ for all $m \in S, A \subset T$ ".

In the examples given for pre S -spaces, (2) (iii), (3) (ii), (4) and all examples given for α -systems in [1] also satisfy (P_2) .

§ 2. Alternate postulates and induced closure systems. The following two theorems have comparison with Theorem 4, [1].

2.1. Theorem. *Let F be a nonvoid family of subsets of T satisfying the following conditions:*

- (i) F is closed under arbitrary intersections
- (ii) $A * s \in F$ whenever $A \in F$ and $s \in S$. Then the operation $A \mapsto \bar{A} = \bigcap \{B \in F \mid A \subset B\}$ makes T into a pre S -space and the family of ideals coincides with F .

Proof. From (i), $A \mapsto \bar{A}$ is a closure operation. From (ii) and Lemma 1.4 continuity axiom is satisfied. Therefore T is a pre S -space with respect to the operation $A \mapsto \bar{A}$ on the subsets of T . To prove that the family of ideals coincides with F , let A be an ideal. Then we have A

$= \bar{A}$, and so A belongs to F by (i). On the other hand let $A \in F$. Then clearly $A = \bar{A}$ which means A is an ideal. This completes the proof.

2.2. Theorem. Let F be a nonvoid family of subsets of T satisfying the following conditions:

- (i) F is closed under arbitrary intersections
- (ii) $A * s \in F$ and $A \subset A * s$ whenever $A \in F$ and $s \in S$. Then the operation $A \mapsto A = \bigcap \{B \in F \mid A \subset B\}$ makes T into a S -space and the family of ideals coincides with F .

Proof. In view of Theorem 2.1 it is enough to show that the ideals in T satisfy (P_2) which is obvious from the second part of (ii).

Remark. Aubert ([1]), in his Theorem 4, assumes that F is closed only under non-empty intersections. But with that hypothesis the theorem is false because the family $\chi = \{\emptyset\}$ satisfies the hypothesis there in, but the conclusion that the family of x -ideals with respect to the operation $A \mapsto A_x$ coincides with χ is not correct since S does not belong to χ . If we modify condition 4.1 as "the intersection of any family of x -ideals is again an x -ideal" then the theorem would be correct, and it is then an easy corollary to our Theorem 2.2.

If T is a pre S -space, then the closure system in T induces a closure system in S satisfying $\overline{MA} \subset \overline{MA}$ for all M, A (see Theorem 2.6). We begin with the following

2.3. Lemma. Let S be a set and let F a subset of $P(S)$, the power set of S . Then $M \mapsto M = \bigcap \{X \in F \mid M \subset X\}$ is a closure operation on S and the resulting closure system in S is the smallest containing F .

Proof. It can be easily seen that $M \subset \overline{M}$ for each $M \subset S$. Let $M, N \subset S$ with $M \subset \overline{N}$. We shall show that $\overline{M} \subset \overline{N}$. By definition $\overline{M} = \bigcap_{\substack{X \in F \\ M \subset X}} X$ and $\overline{N} = \bigcap_{\substack{Y \in F \\ N \subset Y}} Y$. Let $Y \in F$ with $N \subset Y$ (if there is no such Y then $\overline{N} = S$ and $\overline{M} \subset \overline{N}$). Then $\overline{N} \subset Y$ and so $M \subset Y$. Thus $\{Y \in F \mid N \subset Y\} \subset \{Y \in F \mid M \subset Y\}$, and therefore $\overline{M} \subset \overline{N}$. Hence $M \mapsto \overline{M}$ is a closure operation on S . Let $\mathcal{S}$ denote the resulting system of closed sets. Clearly $S \in \mathcal{S}$. Let $\mathcal{T}$ be any closure system in S such that $\mathcal{T} \supset \mathcal{S}$. Now $M \in \mathcal{S}$ implies $M = \overline{M} = \bigcap \{X \in F \mid M \subset X\} \in \mathcal{T}$, showing that $\mathcal{T} \supset \mathcal{S}$. This completes the proof.

Remark. The closure system in S obtained in Lemma 2.3 is nothing but the family of arbitrary intersections of members of F .

Definition. For all A, B we define: $A : B = \{s \in S \mid sB \subset A\}$.

2.4. *Remark.* In the usual notation, the following hold:

(i) $A \subset C$ implies $A : B \subset C : B$. (ii) $M \subset A : B$ iff $MB \subset A$.

(iii) $A : \bigcup_{i \in I} B_i = \bigcap_{i \in I} (A : B_i)$. (iv) $(\bigcap_{i \in I} A_i) : B = \bigcap_{i \in I} (A_i : B)$.

(v) $(\bigcup_{i \in I} A_i) : b = \bigcup_{i \in I} (A_i : b)$.

For any $t \in T$ let $\hat{t}$ denote the mapping from S into T given by $\hat{t}(s) = st$ for all $s \in S$. The following lemma is useful in the sequel.

2.5. *Lemma.* Let T be a pre S -space. Then, for any M ,

$$\begin{aligned} \bigcap \{(\overline{A} : B) \mid A, B \in \mathcal{P}(T) \text{ and } M \subset \overline{A} : B\} &= \bigcap \{(\overline{MA} : A) \mid A \subset T\} = \bigcap \{\overline{Mt} : t \in T\} \\ &= \bigcap \{\overline{A} : t \mid A \subset T, t \in T \text{ and } M \subset \overline{A} : t\} = \bigcap \{\hat{t}^{-1}(\overline{A}) \mid A \subset T, t \in T \text{ and } M \subset \hat{t}^{-1}(\overline{A})\} \end{aligned}$$

Proof. We shall prove, for instance, the first equality and the remaining equalities also can be established by employing similar techniques.

Let F and G respectively denote the families $\{\bar{A}: B \mid M \subset \bar{A}: B, A \subset T, B \subset T\}$, $\{\overline{MA}: A \mid A \subset T\}$ of sets involved in the L.H.S. and R.H.S. of the first equality in question. By (ii) of Remark 2.4 it follows that every member of G is a member of F and hence $\cap F \subset \cap G$. On the other hand let $\bar{A}: B$ be a member of F , i.e. $M \subset \bar{A}: B$, which implies $MB \subset \bar{A}$ and so $\overline{MB} \subset \bar{A}$. Now by (i) of Remark 2.4 we have $\overline{MB}: B \subset \bar{A}: B$. Hence every member of F contains some member of G and so contains $\cap G$ thus showing that $\cap F \supset \cap G$.

2.6. Theorem. *Let T be a pre S -space. Then $M \vdash \bar{M} = \cap \{\overline{MA}: A \mid A \subset T\}$ is a closure operation on S which satisfies*

$$(2.6.1) \quad \overline{MA} \subset \overline{M\bar{A}} \quad \text{for all } M, A.$$

Moreover, the above operation gives the largest closure satisfying (2.6.1).

Proof. By Lemma 2.5 we have $\bar{M} = \cap \{\bar{A}: B \mid A, B \in P(T) \text{ and } M \subset \bar{A}: B\}$ and now, using Lemma 2.3 we get that $M \vdash \bar{M}$ is a closure operation on S . To prove (2.6.1), for any M, A , the continuity axiom gives that $M \subset \overline{MA}: \bar{A}$. Now, since $\bar{M} = \cap \{\bar{A}: B \mid A, B \in P(T) \text{ and } M \subset \bar{A}: B\}$, it follows that $\bar{M} \subset \overline{MA}: \bar{A}$ which gives that $\overline{MA} \subset \overline{M\bar{A}}$. The last part is easy.

2.7. Remark. If $S = T$ and if we assume that T is a commutative groupoid with identity then for any $A \subset T$ the original closure of A (in the pre T -space T) and the induced closure of A as defined above, coincide. In fact, let A_x and A_y respectively denote the two closures of A mentioned in Remark. . For any $B \subset T$ we have $A_x B = BA_x \subset (BA)_x = (AB)_x$ so that $A_x \subset (AB)_x: B$. Therefore $A_x \subset \bigcap_{B \subset T} (AB)_x: B = A_y$. Also if e denotes the identity element of T , $A_y = A_y e \subset A_y e_x \subset (Ae)_x = A_x$. Hence $A_y = A_x$.

2.8. Remark. If the closure in T is the topological closure then the same need not hold for the induced closure in S . To see this, consider the example where $T = \{t_1, t_2, t_3\}$, $S = \{s_1, s_2, s_3\}$ and the S -multiplication in T is given by the table

	t_1	t_2	t_3
s_1	t_1	t_2	t_3
s_2	t_1	t_2	t_2
s_3	t_1	t_1	t_3

Let $F = \{\emptyset, \{t_1\}, \{t_1, t_2\}, T\}$. One can verify that T is a pre S -space with F as the ideal system. Moreover, the members of F constitute the closed sets for a topology in T . But with the induced closure system in S , we get $\overline{\{s_2\}} = \{s_2\}$, $\overline{\{s_3\}} = \{s_3\}$, where as $\overline{\{s_2\} \cup \{s_3\}} = \{s_1, s_2, s_3\}$.

However, (if the closure in T is topological and) if the family $\{A : t \mid A \text{ is closed, } t \in T\}$ is such that the union of any two members again belongs to it, then the induced closure in S is a topological closure and in fact the corresponding topology in S is the smallest such that each $\hat{t}$ ($t \in T$) is continuous. Thus in the case when T is a topological space and S is a family of continuous mappings on T (Vide Example 1 for pre S -spaces), if the above condition is satisfied then the induced closure in S is exactly the one with respect to the pointwise topology (the topology of pointwise convergence cf. Chapter 7 of [3]) for S .

Finally, we have the following theorem concerning the induced ideal system in S .

2.8. Theorem. *Let S be a groupoid, T be a pre S -space such that*

$(ss')t = s(s't)$ for all $s, s' \in S$, and $t \in T$. Then the induced closure system in S satisfies $\overline{MN} \subset \overline{MN}$ for all $M, N \subset S$ and hence makes S into a pre S -space. Further, if T is a S -space then S is also a S -space with the induced closure system.

Proof. Let $M, N \subset S$ and let $s \in \overline{M}$, $s' \in \overline{N}$. Then for any $A \subset T$, $sA \subset \overline{MA} \subset \overline{MA}$ by Theorem 2.6, and similarly $s'A \subset \overline{NA}$. Therefore $ss'A = s(s'A) \subset s(\overline{NA}) \subset \overline{MNA}$ for all A . Hence $ss' \in \overline{MN}$, proving the first part. For the last part let $s \in S$, $M \subset S$. For any $A \subset T$ we have $s\overline{MA} \subset s\overline{MA} \subset \overline{sMA} \subset \overline{MA}$ giving that $s\overline{M} \subset \overline{MA} : A$. Therefore $s\overline{M} \subset \overline{M}$ which completes the proof.

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