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On Groups of Central Type

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Let G be a finite group with center Z . We call G a *group of central type* if there exists an irreducible character χ on G with $\chi^2(1) = |G:Z|$. G. Seitz [2] has characterized all groups having exactly one non-linear irreducible character. In this paper we characterize all groups of central type with exactly two non-linear irreducible characters and also discuss groups with three non-linear irreducible characters.

Lemma [1; p. 31]. *Let G' be the commutator subgroup of G . If $|G'| = p$, a prime, $G' \subset Z$, then every non-linear irreducible character χ satisfies the relation $\chi^2(1) = |G:Z|$.*

Theorem 1. *Let G be a group of central type having exactly two non-linear irreducible characters. Then G is one of the following forms:*

$$(i) \quad |G| = 2^r, r \text{ even}, |G'| = 2, |Z| = 4, G' \subset Z.$$

$$(ii) \quad |G| = 3^m, m \text{ odd}, G' = Z, |G'| = 3.$$

Conversely, if G is of the form (i) or (ii) then G is a group of central type with exactly two non-linear irreducible characters.

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Proof. Let us prove the converse part. If G is of the form (i) or (ii) then by Lemma, the non-linear irreducible characters have degree $|G:Z|^{1/2}$. Let $G = \bigcup_{i=1}^n G'x_i$ where $x_i, \dots, x_n$ are coset representatives of G' in G . Since each coset is a union of conjugacy classes of G , it can be verified that the number of conjugacy classes is precisely $|G:G'| + 2$ by considering the cardinality of Z .

Next we prove the direct part. Let $\chi_1, \dots, \chi_t, \chi_{t+1}, \chi_{t+2}$ be the irreducible characters of G in which χ_{t+1}, χ_{t+2} are non-linear. Let $\chi_{t+2}^2(1) = |G:Z|$. Clearly $1_{G'}^G = \chi_1 + \dots + \chi_t$ where $1_{G'}$ denotes the identity character on G' . Consider 1_Z^G . The following four possibilities can occur.

- (1) $1_Z^G = \chi_1 + \dots + \chi_t$,
- (2) $1_Z^G = \chi_1 + \dots + \chi_\ell$, $\ell < t$,
- (3) $1_Z^G = \chi_1 + \dots + \chi_t + \chi_{t+1}(1)\chi_{t+1}$,
- (4) $1_Z^G = \chi_1 + \dots + \chi_\ell + \chi_{t+1}(1)\chi_{t+1}$, $\ell < t$.

Assume 1). Then $G' = Z$. Let $G = \bigcup_{i=1}^t G'x_i$. Since G has $t+2$ conjugacy classes, we have $|G'| = 3$, and G is of the form (ii).

Assume 2). Let $g \in G'$, $g \notin Z$. Then $1_Z^G = \chi_1 + \dots + \chi_\ell$ gives $(\chi_1 + \dots + \chi_\ell)(g) = 0$. But $(\chi_1 + \dots + \chi_\ell)(g) = \ell$ since $g \in G'$. Hence $G' \subset Z$. Counting of conjugacy classes it results that G is of the form (i).

Assume 3). $1_Z^G = \chi_1 + \dots + \chi_t + \chi_{t+1}(1)\chi_{t+1}$ and $\theta^G = \chi_{t+2}(1)\chi_{t+2}$ for some irreducible character θ on Z . Hence $|Z| = 2$. If $g \in Z$, $g \notin G'$ then $\chi_{t+1}^2(1) = |G:Z|$ a contradiction. Hence $Z \subset G'$. Now G' splits into three conjugacy classes, $1, Z \setminus 1, G' \setminus Z$. Let $x \in G'$ and $0(x) = p \neq 2$. Consider $zx, z \in Z$. $0(zx) = 2p$. But every element of G' is of order of

either 2 or p a contradiction. Hence $|G'| = 2^r$, $r > 1$. Since $G'x_i$ is a single conjugacy class, $|G| = 2^m$, and $\frac{|G|}{|Z|} = \frac{|G|}{|G'|} + \chi_{t+1}^2(1)$. Since $\chi_{t+1}(1) \mid |G|$, we have $\chi_{t+1}(1) = 2^u$ for some u . Hence

$$2^{m-1} = 2^{m-r} + 2^{2u}$$

$$2^{2u} = 2^{m-1} - 2^{m-r} = 2^{m-r}(2^{r-1} - 1).$$

Since $r > 1$, an odd prime divides right hand side a contradiction. Hence 3) cannot occur.

Assume 4). Thus $1_Z^G = \chi_1 + \dots + \chi_\ell + \chi_{t+1}(1)\chi_{t+1}$. Then $\chi_{t+1}^2(1) < |G:Z|$. Let $\theta^G = \chi_{t+2}(1)\chi_{t+2}$. Now $\theta_1^G = \chi_{\ell+1} + \dots + \chi_m$ $m \leq t$ for some $\theta_1 \in \text{irreducible}(Z)$. Clearly $G' \subset Z$. Hence $|G'| = 2$, $|Z| = 4$, and hence by Lemma $\chi_{t+1}^2(1) = |G:Z|$ a contradiction. Therefore if G is a group of central type with exactly two non-linear irreducible characters, then both of them have the same degree $|G:Z|^{1/2}$ and G is of the form (i) or (ii).

Theorem 2. *Let G be a group of central type with exactly 3 non-linear irreducible characters. Then G is of one of the following forms.*

- (1) $|G| = 2^r$, r even, $G' = Z = V_4$ -Klein four group,
- (2) $|G| = 2^r 3^s$, r, s odd, $|G'| = 2$, $|Z| = 6$.

Proof. Let $\chi_1, \dots, \chi_t, \chi_{t+1}, \chi_{t+2}, \chi_{t+3}$ be the irreducible characters of G in which $\chi_{t+1}, \chi_{t+2}, \chi_{t+3}$ are non-linear. Let $G = \bigcup_{i=1}^t G'x_i$. Then G has $t+3$ conjugacy classes. Let $\chi_{t+3}^2(1) = |G:Z|$. The following possibilities can occur.

- (1) $1_Z^G = \chi_1 + \dots + \chi_t$
- (2) $1_Z^G = \chi_1 + \dots + \chi_\ell$, $\ell < t$.
- (3) $1_Z^G = \chi_1 + \dots + \chi_t + \chi_{t+1}(1)\chi_{t+1}$

$$(4) \quad 1_Z^G = \chi_1 + \cdots + \chi_\ell + \chi_{t+1}(1)\chi_{t+1}, \quad \ell < t.$$

$$(5) \quad 1_Z^G = \chi_1 + \cdots + \chi_t + \chi_{t+1}(1)\chi_{t+1} + \chi_{t+2}(1)\chi_{t+2}$$

$$(6) \quad 1_Z^G = \chi_1 + \cdots + \chi_\ell + \chi_{t+1}(1)\chi_{t+1} + \chi_{t+2}(1)\chi_{t+2}, \quad \ell < t.$$

Assume 1). Then $G' = Z$. From the coset decomposition $G = \bigcup_{i=1}^t Gx_i$ and

counting of conjugacy classes it follows that $|G'| = 2$ or 4 . If $|G'| = 2$, then $|Z| = 6$ and G is of the form (2). If $|G'| = 4$ then G is of the form (1).

Assume 2). $1_Z^G = \chi_1 + \cdots + \chi_\ell$, $\ell < t$ implies $G' \subset Z$ and further $|G'| = 2$, $|Z| = 6$ and clearly G is of the form (2).

Assume 3). Then $\chi_{t+1}^2(1) < |G:Z|$ and $Z \subset G'$. Since $\theta^G = \chi_{t+3}(1)\chi_{t+3}$ we have $\theta \in \text{irreducible}(Z)$. Since $\sum_{i=1}^{t+3} \chi_i^2(1) = |G|$, we have $3 \frac{|G|}{|Z|} = |G|$ and so $|Z| = 3$. Let $Z = \{1, z, z^2\}$. Now G' splits into 4 conjugacy classes $1, z, z^2, G' \setminus Z$. Every element of $g \in G' \setminus Z$ is of order p where p is a prime $\neq 3$ say. Consider gz . Then $0(gz) = 3p$ a contradiction. Hence every element of G' is of order a power of 3. Hence $|G'| = 3^r$, $r > 1$. Since each coset $G'x_i$ is a single conjugacy class, 3 is the only prime dividing $|G|$. Hence $|G| = 3^n$. Now $\chi_{t+2}^2(1) = 3^{2t}$ and so $3^{n-1} = 3^{n-r} + 3^{2t}$ and $3^{2t} = 3^{n-r}(3^{r-1} - 1)$. Since $r > 1$, right hand side is even, a contradiction.

Assume 4). Thus $1_Z^G = \chi_1 + \cdots + \chi_\ell + \chi_{t+1}(1)\chi_{t+1}$, $\ell < t$, then $\chi_{t+1}^2(1) < |G:Z|$ and $\chi_{t+1}/Z = \chi_{t+1}(1)1$. Let $\theta^G = \chi_{t+3}(1)\chi_{t+3}$. If $\theta_1^G = \chi_{\ell+1} + \cdots + \chi_r$, $\theta \in \text{irreducible}(Z)$ then $G' \subset Z$ and so $\chi_{t+1}/_{G'} = \chi_{t+1}(1)1$, a contradiction. Thus $\theta_1^G = \chi_{\ell+1} + \cdots + \chi_t + \chi_{t+2}(1)\chi_{t+2}$. Hence $|Z| = 3$. If $g \in Z$, $g \neq 1$ then

$$\frac{|G|}{|Z|} \{1 + \theta(g)\} = t_1^2 + t_2^2$$

i.e.,

$$\frac{|G|}{|Z|} (1 + w) = t_1^2 + t_2^2$$

where w is a cube root of unity ($\neq 1$). Left hand side is a complex number and right hand side is a real number a contradiction. Hence $Z \subset G'$. $|Z| = 3$. As in the previous case, $1_Z^G = \chi_1 + \dots + \chi_\ell + \chi_{t+1}(1)\chi_{t+1}$ cannot occur.

Assume 5). $1_Z^G = \chi_1 + \dots + \chi_t + \chi_{t+1}(1)\chi_{t+1} + \chi_{t+2}(1)\chi_{t+2} + \chi_{t+3}(1)\chi_{t+3}$, $\theta \in \text{Irr}(Z)$. Clearly $|Z| = 2$ and $Z \subset G'$. Let G' split into 3 conjugacy classes, $1, z, G' \setminus Z$. If every element x of $G' \setminus Z$ is of order $p \neq 2$, then $0(xz) = 2p$ a contradiction. Hence $|G'| = 2^r$. Further consider $G'x_2$. If it splits into two conjugacy classes and if $0(x_2) = q$ then consider $G'x_2^2$. $0(x_2^2) = q$ and so $0(x_2^2z) = q$ a contradiction, since $G'x_2^2$ is a single conjugacy class. Therefore $|G| = 2^m$.

Now

$$\begin{aligned} 2^{m-1} &= 2^{m-r} + 2^{2t_1} + 2^{2t_2}, \\ 2^{2t_1} + 2^{2t_2} &= 2^{m-r}(2^{r-1} - 1). \end{aligned}$$

Let $t_1 < t_2$. Then

$$2^{2t_1}(1 + 2^{2t_2-2t_1}) = 2^{m-r}(2^{r-1} - 1).$$

This forces

$$1 + 2^{2t_2-2t_1} = 2^{r-1} - 1.$$

$$2 + 2^{2t_2-2t_1} = 2^{r-1},$$

$$2(1 + 2^{2t_2-2t_1-1}) = 2^{r-1},$$

i.e., $r - 1 = 2$ and $2t_2 - 2t_1 - 1 = 0$.

i.e., $(t_2 - t_1) = \frac{1}{2}$, a contradiction. Similarly so if $t_2 < t_1$ and hence $t_1 = t_2$.

$$2^{2t_1+1} = 2^{m-r}(2^{r-1} - 1)$$

an odd prime divides right hand side, a contradiction. Let G' splits into four conjugacy classes $1, z, C_1, C_2$. Then either $|G'| = 2^r$ or

$|G'| = 2^n p^8$. Further, since $G'x_i$ is a single conjugacy class, $|G| = 2^m p^n$. Since

$$(1) \quad \frac{|G|}{|Z|} - \frac{|G|}{|G'|} = t_1^2 + t_2^2$$

and $t_1, t_2/|G'|$ it can be checked that there exists no solution for (1).

Assume 6). $1_Z^G = \chi_1 + \dots + \chi_\ell + \chi_{t+1}(1)\chi_{t+1} + \chi_{t+2}(1)\chi_{t+2}$. Then $\chi_{t+1}^2(1) < |G:Z|$, $\theta^G = \chi_{t+3}(1)\chi_{t+3}$. $\theta_1^G = \chi_{\ell+1} + \dots + \chi_m$, $m \leq t$, $\theta_1 \in \text{Irr}(Z)$. This implies $G' \subset Z$ and $|G'| = 2$, $|Z| = 6$. By Lemma $\chi_{t+1}^2(1) = \chi_{t+2}^2(1) = \chi_{t+3}^2(1) = |G:Z|$ a contradiction.

Remark. If G is of the form (2), then it can be checked that G has exactly $|G:G'| + 3$ conjugacy classes and by Lemma, G is of the central type. But if G is of the form (1), then G need not contain exactly 3 non-linear irreducible characters. For example consider $G: D_8 \times D_8$, where D_8 is dihedral group of order 8. $G' = Z = V_4$. But G has 9 non-linear irreducible characters.

References

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