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0-Ideals in Semilattices

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Introduction. In this paper, we extend some of the results of Balachandran [2] and Venkatanarasimhan [11] to 0-distributive semilattices. Some equivalent conditions are given for a 0-distributive semilattice to be a Boolean algebra. Using these we prove that a 0-distributive semilattice is a finite Boolean algebra if and only if every ideal of L is a non dense 0-ideal. Further we generalize some of the results of Adams [1] to 0-distributive semilattices.

For convenience sake, this paper is divided into two sections. In the first section we collect some known results which we need in this paper and Section 2 deals with 0-ideals in 0-distributive semilattices.

§ 1. Preliminaries. Through out this paper we shall be concerned with meet semilattices. We use the symbols $\vee$ (join) and $\wedge$ (meet) to indicate the least upper bound and the greatest lower bound of a subset when they exist. The least (greatest) element is denoted by $0(1)$, when it exists.

We follow the definitions given in [10]. A *filter* of a semilattice

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F is a non empty subset of L such that $x \wedge y \in F$ if and only if $x \in F$ and $y \in F$. The principal filter generated by an element a of L , i.e., the set $\{x \in L \mid x \geq a\}$, will be denoted by $[a]$. A filter F of L is *prime* if, whenever two filters F_1 and F_2 are such that $\emptyset \neq F_1 \cap F_2 \subset F$, F_1 or F_2 belongs to F . A proper filter F is *maximal* if the only filter strictly containing F is L . When L has the least element 0 , a proper filter F is maximal if and only if for any $a \notin F$, there exists an element $b \in F$ such that $a \wedge b = 0$.

An *ideal* I of a semilattice L is a non empty subset of L such that

$$(I_1) \quad y \leq x \text{ and } x \in I \text{ imply } y \in I,$$

$$(I_2) \quad \text{for any } x, y \in I, \text{ there exists } z \in I \text{ such that } z \geq x \text{ and } z \geq y.$$

The principal ideal generated by an element a of L , i.e., the set $\{x \in L \mid x \leq a\}$ will be denoted by $\langle a \rangle$. The notions of prime and maximal ideals can be defined in the same way as for filters. A proper ideal I of L is *minimal prime* if it contains no other prime ideal. An ideal I of a semilattice L with 0 is said to be *non dense* if $I^* \neq \{0\}$, where $I^* = \{x \in L \mid x \wedge y = 0 \text{ for all } y \in I\}$.

A semilattice L is said to be *up-directed* if, for any $x, y \in L$ there exists $z \in L$ such that $z \geq x$ and $z \geq y$. Let us remark that in an up-directed semilattice L , a filter F is prime if and only if $L-F$ (set theoretic complement of F) is an ideal of L .

The concept of 0-distributive semilattice is due to J. Varlet [10]. A semilattice L with 0 is *0-distributive* if, for any $a \in L$, the subset $(a)^* = \{x \in L \mid x \wedge a = 0\}$ is an ideal. Dually a semilattice L with 1 is

1-distributive if $a \vee b, a \vee c$ ($a, b, c \in L$) exist and equal 1, then $a \vee (b \wedge c)$ exists and is equal to 1. Let us notice that a 0-distributive semilattice is an up-directed semilattice. In [10] Varlet has proved that an up-directed semilattice L with 0 is 0-distributive if and only if any maximal filter of L is prime. This result shows that minimal prime ideals exist in 0-distributive semilattices and they are precisely the set theoretic complements of maximal filters. It can also be observed that a prime ideal P of a 0-distributive semilattice L is minimal prime if and only if for any $x \in P$, there is $y \notin P$ ($y \in L$) such that $x \wedge y = 0$.

A semilattice L with 0 is *semicomplemented* if for any $x \in L$ ($x \neq 1$, if 1 exists in L), there exists $y \in L$ such that $x \wedge y = 0$. A semilattice L with 0 is *weakly complemented* if, for any pair a, b ($a < b$) of elements of L , there is an element c such that $a \wedge c = 0$ and $b \wedge c \neq 0$. A semilattice L with 0 and 1 is said to be *complemented* if, for any $a \in L$, there is $b \in L$ such that $a \wedge b = 0$ and 1 is the only upper bound of a and b (b is called a *complement* of a).

§ 2. 0-ideals in semilattices. Throughout this section, L will denote a 0-distributive semilattice. For any ideal I of L , we denote $I^* = \{x \in L \mid x \wedge y = 0 \text{ for all } y \in I\}$. An element a of L is said to be *dense* if $(a)^* = \{0\}$. For any filter F of L , we denote $0(F) = \{x \in L \mid x \wedge y = 0 \text{ for some } y \in F\}$. All ideals (filters) of L are assumed to be proper. The set theoretic complement of any subset A of L is denoted by $L-A$ and the empty set is denoted by $\emptyset$. Set inclusion, set union and set intersection are denoted by $\subset, \cup$ and $\cap$, respectively.

We shall begin with the following definition.

Definition. An ideal I of L is said to be an *0-ideal* if $I = 0(F)$ for some proper filter F of L .

It may be mentioned that our definition of 0-ideal coincides with that of Cornish [4] in distributive lattices (even in 0-distributive lattices). It should also be noted that 0-ideals are not new. They are precisely the so called last-residue class studied in ([2], [5], [6], [7] and [11]).

From the definition of 0-ideal, it can easily be seen that minimal prime ideals of L and the ideals of the form $(\alpha)^*$ ($0 \neq \alpha \in L$) are examples of 0-ideals. It can also be observed that the intersection of any two 0-ideals of L is again an 0-ideal. A detailed investigation on 0-ideals will be found in [2] and [7].

The following theorem is a generalization of the results of Balachandran ([2], Theorems 5, 6 and 7) and Venkatanarasimhan ([11], Theorem 7.2.5 and Theorem 7.2.9). The proof of the theorem involves only minor changes in the proofs of the corresponding known results (Theorems 5, 6 and 7 in [2] and Theorem 7.2.5 and Theorem 7.2.9 in [11]) and hence it is left to the reader.

Theorem 1. *Let I be an ideal and F be a filter of L . If G is the intersection of all maximal filters containing F , then the following statements are equivalent:*

- (i) $I = 0(F)$.
- (ii) $I = 0(G)$ and $I = 0(J)$ imply $J \subset G$ for any filter J of L .
- (iii) $I = \bigcup_{x \in F} (x)^*$.
- (iv) $I = \bigcap \{P \mid P \text{ is a minimal prime ideal of } L, P \cap F = \emptyset\}$.

As a consequence of Theorem 1, we have the following corollaries:

Corollary 1. A necessary and sufficient condition for an ideal I of L to be contained in a minimal prime ideal is that I is contained in some 0-ideal.

Corollary 2. A prime ideal P of L is a minimal prime ideal if and only if P is an 0-ideal.

Corollary 3. A filter F of L is a maximal filter if and only if $L-F$ is an 0-ideal.

Corollary 4. A non dense ideal I of L is a minimal prime ideal if and only if I satisfies the following condition $(\bullet)$: For any $x \in L$, $x \notin I$ implies $I^* \cap (x)^* = \{0\}$.

Proof. Obviously every minimal prime ideal of L satisfies the condition $(\bullet)$. Let us now assume that I satisfies the condition $(\bullet)$. We first show that I is prime. Observe that by hypothesis and the non dense property of I , we have $I = I^{**}$. Let $a \wedge b \in I$ and $a \notin I$ for any two elements a and b of L . Then by condition $(\bullet)$, $I^* \cap (a)^* = \{0\}$. Also for any $c \in I^*$, $c \wedge (a \wedge b) = 0$, so that $c \wedge b \in I^* \cap (a)^*$ and hence $c \wedge b = 0$. Consequently $b \in I^{**} = I$. Therefore I is prime. Again since I is prime and non dense, it follows that for any $x \in I$, there is $y \notin I$ such that $x \wedge y = 0$. Hence I is a minimal prime ideal.

Remark. From the definition of 0-ideal, it can easily be seen that L is dense (that is $[0]$ is a prime ideal of L) if and only if every 0-ideal of L is prime. Also from Corollary 1, every 0-ideal of L is non dense if and only if every minimal prime ideal of L is non dense.

Theorem 2. *An ideal I of L is an 0-ideal if and only if I is the set intersection of a family of minimal prime ideals which are such that their set union does not contain any minimal prime ideal not in the family.*

Proof. Similar to Theorem 8, [2] (also see [11], Theorem 7.2.11).

We now state a lemma without proof which we need.

Lemma 1. *If L is weakly complemented and d is a dense element of L , then d is the greatest element of L .*

Our next theorem is a generalization of the results of G. Grätzer and E. T. Schmidt ([5], Section 3, Theorem 4) and K. Iséki ([6], Theorem 11).

Theorem 3. *The following statements are equivalent:*

- (i) L is a Boolean algebra.
- (ii) Every ideal of L is an 0-ideal.
- (iii) L is weakly complemented and for each $x \in L$, $(x)^{**} = (y)^*$ for some $y \in L$.

Proof. (i) $\Rightarrow$ (ii) is obvious. (ii) $\Rightarrow$ (iii). Suppose (ii) holds. Let $a < b$ for some $a, b \in L$. By hypothesis, $(a] = 0(F)$ for some filter F of L . So $a \wedge c = 0$ for some $c \in F$. Obviously $b \wedge c \neq 0$. Therefore L is weakly complemented. Now let $x \in L$. Once again by hypothesis, $(x] = 0(F)$ for some filter F of L , so that $x \wedge y = 0$ for some $y \in F$. Now it can be easily seen that $(x)^{**} = (y)^*$. Thus (iii) holds.

(iii) $\Rightarrow$ (i). Suppose (iii) holds. We first show that L is complemented. Let a be any element of L . Then by (iii) $(a)^{**} = (b)^*$ for some $b \in L$. Clearly $a \wedge b = 0$. As L is up-directed, there exists $c \in L$ such

that $c \geq a$ and $c \geq b$. Now it can be easily seen that c is a dense element of L so that by Lemma 1, c is the greatest element of L . This is true for any upper bound of a and b . Therefore $a \vee b$ exists and is equal to the greatest element of L . That is b is a complement of a . Hence L is complemented. Again since L is weakly complemented, 0-distributive and complemented, it follows that L is uniquely complemented. Also for any pair $x, y \in L$, $x \leq y$ implies $y' \leq x'$ ($'$ denotes complementation). Consequently, for any pair $x, y \in L$, $x \vee y$ exists and is equal to $(x' \wedge y')'$. Therefore L is a lattice and hence (by Corollary 2, [9]: If a 0-distributive lattice L is complemented and weakly complemented, then L is Boolean) L is a Boolean algebra. This completes the proof of Theorem 3.

We now give a necessary and sufficient condition for a 0-distributive semilattice to be a finite Boolean algebra.

Theorem 4. *L is a finite Boolean algebra if and only if every ideal of L is a non dense 0-ideal.*

Proof. The "only if" part is obvious. We now establish the "if" part. We assume that every ideal of L is a non dense 0-ideal. Observe that by Theorem 3, L is a Boolean algebra and also every prime ideal of L is non dense. So if P is any prime ideal of L , then $P = (x']$ for some $0 \neq x \in P^*$ (x' is the complement of x). That is, every prime ideal of L is principal. Therefore (by Theorem 1, [3]: Let L be a distributive lattice with 1. If every prime ideal in L is principal, then every ideal is principal) every ideal of L is principal. Consequently L is a finite Boolean algebra. This completes the proof of Theorem 4.

Definition 8. An element x of a semilattice S is said to be *semi compact* if for any $A \subseteq S$, $[x]$ is the smallest ideal containing A , then there exists a finite subset B of A such that $[x]$ is the smallest ideal containing B .

Let us remark that every element in a lattice is semi compact. A detailed investigation on "semi compactness" will be found in [8].

We now prove some lemmas which we need in further results.

Lemma 2. *If L is a bounded semi complemented semilattice, then L is 1-distributive.*

Proof. Suppose $x \vee y, x \vee z$ ($x, y, z \in L$) exist and are equal to 1. Let $x \leq w$ and $y \wedge z \leq w$ for some $w \in L$. If possible assume that $w \neq 1$. As L is semi complemented, $w \wedge a = 0$ for some $0 \neq a \in L$. Clearly $x \wedge a = 0 = (y \wedge z) \wedge a$. Since L is 0-distributive, we have $(a \wedge y) \wedge (x \vee z) = 0$ and so $a \wedge y = 0$. Again $0 = a \wedge (x \vee y) = a \wedge 1$ and therefore $a = 0$, a contradiction. Hence 1 is the only upper bound of x and $(y \wedge z)$. This shows that L is 1-distributive.

Lemma 3. *If L is a bounded semi complemented semilattice such that 1 is semi compact, then every maximal ideal of L is prime.*

Proof. Let M be a maximal ideal of L and let $x, y \notin M$. Then $[1]$ is the smallest ideal containing $\{M, x\}$. Since 1 is semi compact, there exist $t_1, t_2, \dots, t_n \in M$ such that $[1]$ is the smallest ideal containing $\{t_1, t_2, \dots, t_n, x\}$. As M is an ideal, there is $t \in M$ such that $t \geq t_1, t_2, \dots, t_n$. Clearly $t \vee x$ exists and is equal to 1. Similarly $w \vee y = 1$ for some $w \in M$. Choose $z \in M$ such that $z \geq t, w$; so

that $z \vee x$ and $z \vee y$ exist and are equal to 1. Consequently by Lemma 2, $z \vee (x \wedge y)$ exists and is equal to 1. Again since $z \in M$ and $z \vee (x \wedge y) = 1$, it follows that $x \wedge y \notin M$. Therefore M is a prime ideal.

Our next theorem is a generalization of a result of Adams ([1], Theorem 1).

Theorem 5. *If L is a bounded semilattice such that 1 is semi compact, then the following conditions are equivalent:*

- (i) L is complemented
- (ii) An ideal of L is prime if and only if it is maximal
- (iii) Every maximal ideal of L is an 0-ideal

Proof. (i) $\Rightarrow$ (ii). Suppose (i) holds. Let P be a prime ideal of L . If P is strictly contained in some ideal M of L , then there exists $x \in M$ such that $x \notin P$; so that the complement x' of x must belong to P and hence belong to M . Consequently $M = L$. Therefore P is maximal. Conversely, by (i), L is semi complemented and so by Lemma 2, every maximal ideal is prime.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. Let M be a maximal ideal of L . Then by (ii), $L-M$ is a proper filter and also $0(L-M) \subset M$. Again by (ii), $L-M$ is a maximal filter and so $M \subset 0(L-M)$. Therefore $M = 0(L-M)$. This shows that M is an 0-ideal.

(iii) $\Rightarrow$ (i). Suppose (iii) holds. Let $a \in L$. Observe that by (iii) and Corollary 1, every maximal ideal is a minimal prime and so no maximal ideal contains both a and $(a)^*$. Therefore $\{1\}$ is the smallest ideal containing a , $(a)^*$. Since 1 is semi compact, there exists $b_1, b_2, \dots, b_n \in (a)^*$ such that $\{1\}$ is the smallest ideal containing $\{a, b_1, b_2,$

$\dots, b_n\}$. As $(a)^*$ is an ideal, there is $b \in (a)^*$ such that $b \geq b_1, b_2, \dots, b_n$. Now it can be easily seen that $a \vee b$ exists and is equal to 1. Clearly $a \wedge b = 0$. Therefore b is a complement of a . This shows that L is complemented. Thus (i) holds. This completes the proof of Theorem 5.

The following theorem is an extension of a result of Adams ([1], Theorem 2):

Theorem 6. If L is a bounded weakly complemented semilattice such that 1 is semi compact, then following statements are equivalent:

- (i) *L is a Boolean algebra*
- (ii) *An ideal of L is prime if and only if it is maximal*
- (iii) *Every maximal ideal of L is an 0-ideal.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) follows from Theorem 5.

(iii) $\Rightarrow$ (i). Suppose (iii) holds. Then by Theorem 5, L is complemented. Since L is weakly complemented, 0-distributive and complemented, it follows that L is a Boolean algebra (for proof see second para in the proof of Theorem 3, (iii) $\Rightarrow$ (i)). Thus (i) holds. This completes the proof of Theorem 6.

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