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Cornish, H. William

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On Positive Implicative BCK-algebras

By William H. CORNISH

0. Introduction. In [6; Theorem 2], Iséki and Tanaka gave a description of the ideal generated by a subset of a BCK-algebra. From this and the BCK-identities $(x * y) * z = (x * z) * y$ and $0 * x = 0$, it follows that an element d of a BCK-algebra $(A; *, 0)$ is in the ideal $\langle b_1, \ldots, b_t \rangle$ generated by $b_1, \ldots, b_t \in A$ if and only if there are integers $m_1, \ldots, m_t \geq 1$ such that

$(\ldots(\ldots(\ldots(\ldots(\ldots(d * b_{11}) * \ldots) * b_{1m_1}) * \ldots) * b_{t1}) * \ldots) * b_{tm_t} = 0$, where $b_{i_1} = \ldots = b_{im_i} = b_i$ for each $i = 1, \ldots, t$. In Section 1, we show that a BCK-algebra $(A; *, 0)$ is positive implicative, i.e. satisfies the identity $(x * y) * z = (x * z) * (y * z)$, when and only when the above formula simplifies to $\langle b_1, \ldots, b_t \rangle = \{d \in A: (\ldots(d * b_1) * \ldots) * b_t = 0$ for any $b_1, \ldots, b_t \in A\}$. The proof of this result is interesting as it shows that the upper semilattice $F(A)$ of finitely generated ideals of a positive implicative BCK-algebra $(A; *, 0)$ is a dual relatively pseudocomplemented semilattice, or equivalently a positive implicative BCK-algebra satisfying Iseki's condition(S), and the map $\eta_A: (A; *, 0) \to (F(A); *, 0)$, where $\eta_A(a) = \langle a \rangle$ for each $a \in A$, is a BCK-embedding. From this, it follows that the variety V of positive

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implicative BCK-algebras with condition (S) forms a full reflective subcategory of the variety U of positive implicative BCK-algebras and their BCK-homomorphisms; η: id $\mathsf{U} \xrightarrow{\cdot} F$, where η_A is the component of η at $(A; *, 0) \in \mathsf{U}$, is the front adjunction to the reflector F. Another consequence is a nice description of the principal congruence $\Theta(a, b)$ which identifies a and b in a positive implicative algebra $(A; *, 0)$, and this description characterizes U amongst classes of BCK-algebras which are closed under the formation of homomorphic images. It also shows that U is a congruence-distributive variety. We discuss this in Section 2. There, we also give both an improvement and an extension. The variety of BCK-algebras satisfying the identity $(x * (x * y)) * (y * x) = (y * (y * x)) * (x * y)$ contains the variety U, due to Iséki and Tanaka [7; Proposition 5, p.14]. We show that this apparently larger variety is congruence-3-distributive by demonstrating that $t_1(x, y, z) = (x * ((x * y) * (z * y))) * ((y * x) * (y * z))$ and $t_2(x, y, z) = (z * ((y * x) * (y * z))) * ((x * y) * (z * y))$ are suitable polynomials for an application of Jónsson's Theorem [8; Theorem 2.1] in the case where $n = 3$.

1. Finitely generated ideals. When it comes to BCK-algebras, we recommend Iséki and Tanaka [7] for general theory, Iséki and Tanaka [6] for ideal-theory, and Iséki [4] for his condition (S). The notation will follow that of these papers and Section 0. In [13; Theorem 2], Yutani showed the class of positive implicative BCK-algebras is equationally definable. Iséki examined the class of positive implicative BCK-algebras with condition (S) in [4; Section 4] and showed that it is a variety in [5]. Due to Iséki's work, it can be seen that a positive implicative BCK-

algebra is precisely the groupoid-dual, of a positive implication algebra in the sense of Rasiowa [11; Chapter 2, Sections 2-4].

Let $(A; \vee)$ be an upper semilattice. Then, A is said to be (*dual*) *relatively pseudocomplemented* if, for each $a, b \in A$, the subset $\{c \in A: a \leq b \vee c\}$ has a smallest element which is denoted by $a * b$. Such a semilattice necessarily has a smallest element 0; $0 = a * a$ for any $a \in A$. Our notation for the relative pseudocomplement has been chosen to be consistent with BCK-theory; Schmidt [12] uses a notation which is the groupoid-dual of ours. Relatively pseudocomplemented *lower* semilattices have been examined in detail by Nemitz [9].

Recall that a BCK-algebra $(A; *, 0)$ satisfies *condition (S)* if, for any, $a, b \in A$, the subset $\{c \in A: c * a \leq b\}$ has a largest element which is denoted by $a \circ b$. Recently, Yutani [15, Proposition 1] has characterized the binary operation which results from the condition (S): a BCK-algebra $(A; *, 0)$ has condition (S) if and only if it possesses a binary operation, $\circ$, such that for all $x, y, z \in A$,

(1.0) $x * (y \circ z) = (x * y) * z$.

The next lemma must be part of the folk-lore; we include it for the sake of clarity.

Lemma 1.1. *An algebra* $(A; *, \circ, 0)$ *of type* $(2, 2, 0)$ *is a positive implicative* BCK*-algebra with condition (S) if and only if* $(A; \circ, 0)$ *is an (dual) relatively pseudocomplemented upper semilattice with smallest element* 0, *wherein* $a * b$ *is the (dual) relative pseudocomplement of* $a, b \in A$, *i.e. the smallest element of* $\{c \in A: a \leq b \circ c\}$.

Proof. The assertion about the semilattice associated with a positive implicative BCK-algebra with condition (S) follows immediately from Iséki [4; Theorem 9].

Conversely, suppose $(A; \vee, *, 0)$, of type $(2, 2, 0)$, is a relatively pseudocomplemented upper semilattice. Of course, $a \leq b$ (i.e. $b = a \vee b$) if and only if $a * b = 0$. Hence, the order will be consistent with the BCK-order once it has been shown that $(A; *, 0)$ is a BCK-algebra. In other words, the BCK-axioms (i) $0 \leq x$ (ii) $x \leq x$ (iii) $x \leq y$ and $y \leq x$ imply $x = y$, and (iv) $x \leq y$ iff $x * y = 0$, are automatically satisfied. As $x \leq y \vee (x * y)$, $x \leq (x * y) \vee y$, and so the BCK-axiom, (v) $x * (x * y) \leq y$, is satisfied.

As $(z * y) \vee y \geq z$ and $z \vee (x * z) \geq x$, we have: $y \vee ((x * z) \vee (z * y))$ $= (y \vee (z * y)) \vee (x * z) \geq z \vee (x * z) \geq x$. Hence, $x * y \leq (x * z) \vee (z * y)$, and consequently the BCK-axiom, (vi) $(x * y) * (x * z) \leq z * y$, holds. Thus, $(A; *, 0)$ is a BCK-algebra; it clearly has condition (S), whereby $x \circ y = x \vee y$. Finally, it is positive implicative as it satisfies (vii) $(x * y) * z = (x * z) * (y * z)$. This has been established in dual form for lower semilattice by Nemitz in [9; Lemma 2.1 (vii)]; we can, therefore, omit the proof.

From now on, the two points of view of 1.1 will be used with little or no comment.

Recall that an *ideal* of a BCK-algebra $(A, *, 0)$ is a subset K of A such that (i) $0 \in K$, and (ii) $x \in K$ whenever $x, y \in A$ are such that $x * y, y \in K$. The set $\mathcal{J}(A)$ of all ideals is an algebraic closure

system on A and so forms a complete algebraic lattice. The supremum of two ideals K_1 and K_2 is denoted by $K_1 \vee K_2$. Of course, $K_1 \vee K_2 = (K_1 \cup K_2) = \cap \{K \in J(A): K_1 \cup K_2 \subset K\}$ —— the ideal generated by $K_1 \cup K_2$. The infimum in the lattice $J(A)$ is set-intersection. We use $F(A)$ to denote the set of finitely generated ideals; $(F(A); \vee)$ is a subsemilattice of the upper semilattice $(J(A); \vee)$. A subset K of A is called an *implicative ideal* if (i) $0 \in K$, and (ii) $y * z \in K$, whenever $x, y, z \in A$ are such that $(y * x) * z, x * z \in K$; an implicatice ideal is an ideal, [6; Theorem 4]. In what follows, we need the following equivalences; see Iséki [4; Section 4] and Iséki and Tanaka [6; Section 4].

Lemma 1.2. *The following conditions on a* BCK-*algebra* $(A; *, 0)$ *are equivalent.*

(i) A *is positive implicative.*

(ii) *For each* $K \in J(A)$ *and* $a \in A$, $\{b \in A: b * a \in K\}$ *is an ideal, and consequently the supremum* $K \vee (a)$ *in the lattice* $J(A)$.

(iii) *Each ideal of* A *is an implicative ideal.*

Theorem 1.3. *The following conditions on a* BCK-*algebra* $(A; *, 0)$ *are equivalent.*

(i) A *is positive implicative.*

(ii) *For any* $b_1, \ldots, b_t \in A$, *the finitely generated ideal* $(b_1, \ldots, b_t)$ *is* $\{a \in A: (\ldots(a * b_1) * \ldots) * b_t = 0\}$.

(iii) *The upper semilattice* $F(A)$ *of all finitely generated ideals is (dual) relatively pseudocomplemented and the natural map* $\eta_A: A \longrightarrow F(A)$, *given by* $\eta_A(a) = (a)$ *for each* $a \in A$, *is a* BCK-*embedding of the underlying* BCK-*algebras.*

Proof. (i) $\Rightarrow$ (ii). Suppose A is positive implicative. Then, $\langle b_1\rangle$ is the order-ideal $(b_1] = \{a \in A: a \leq b_1$, i.e. $a * b_1 = 0\}$, c.f. Lemma 1.2 and Iséki [4; Lemma 3, p. 115]. In other words, (ii) holds when $t = 1$. Using (ii) of Lemma 1.2, it is now easy to complete the proof by induction.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. Let $K_1 = \langle a_1, \ldots, a_n\rangle$, $K_2 = \langle b_1, \ldots, b_m\rangle$ be two finitely generated ideals.

Let $K_1 * K_2$ be the finitely generated ideal $c_1, \ldots, c_n$, where for $i = 1, \ldots, n$, $c_i = (\ldots.(a_i * b_1) * \ldots.) * b_m$. We proceed to show that $K_1 * K_2$ is the relative pseudocomplement. (Then, $K_1 * K_2$ is defined independently of the generating systems of K_1 and K_2).

Since, $((\ldots.(a_i * b_1) * \ldots.) * b_m) * c_i = c_i * c_i = 0$ for each $i = 1, \ldots, n$, $K_1 = \langle a_1, \ldots, a_n\rangle \subseteq \langle b_1, \ldots, b_m\rangle \vee \langle c_1, \ldots, c_n\rangle = K_2 \vee K_1 * K_2$. Conversely, suppose $K_3 = \langle d_1, \ldots, d_r\rangle$ is such that $K_1 \subseteq K_2 \vee K_3$. Because of (ii), we have, for each $i = 1, \ldots, n$,

$$(\ldots.(\ldots.(\ldots.(\ldots.(a_i * b_1) * \ldots.) * b_m) * d_1) * \ldots.*) * d_r = 0.$$

But this says that $c_i \in \langle d_1, \ldots, d_r\rangle$ for each $i = 1, \ldots, n$. In other terms, $K_1 * K_2 \subseteq K_3$, and so $F(A)$ is relatively pseudocomplemented. From the above formula, we also have $\langle a * b\rangle = \langle a\rangle * \langle b\rangle$ for any $a, b \in A$. That is, η_A is a BCK-homomorphism. But, again by (ii), $\langle a\rangle = (a]$ $= \{a \in A: c \leq a\}$. Hence, η_A is one-to-one.

(iii) $\Rightarrow$ (i) is an easy consequence of Lemma 1.1.

Lemma 1.4. *Let* $(A: *, 0)$ *be a positive implicative* BCK*-algebra,* $(B; \vee)$ *be a relatively pseudocomplemented upper semilattice, and* $f: A \to B$ *be a map which is a homorphism of the* BCK*-algebra* $(A; *, 0)$. *into the*

associated BCK-*algebra* $(B; *, 0)$. *Define* $\bar{f}$: $F(A) \longrightarrow B$ *by* $\bar{f}(K) = f(a_1) \vee \ldots\ldots \vee f(a_n)$, *when* $K = \langle a_1, \ldots., a_n\rangle \in F(A)$ *with* $a_1, \ldots., a_n \in A$. *Then,* $\bar{f}$ *is a homomorphism of the* BCK-*algebra* $(F(A); *, \vee, 0)$ *with condition*(S) *into the* BCK-*algebra* $(B; *, \vee, 0)$ *with condition*(S). *Moreover,* $f = \bar{f}\eta_A$ *and* $\bar{f}$ *is uniquely determined by these conditions. In addition,* $\bar{f}$ *is one-to-one if and only if* f *is one-to-one.*

Proof. Firstly, we demonstrate that $\bar{f}$ is well-defined. Towards this goal, suppose $K \in F(A)$ and $K = \langle a_1, \ldots., a_n\rangle = \langle b_1, \ldots., b_m\rangle$. Then, for any $i = 1, \ldots., n$, $a_i \in \langle b_1, \ldots., b_m\rangle$ and so, by Theorem 1.3, $(\ldots..(a_i * b_1) * \ldots.) * b_m = 0$. Hence, $(\ldots..(f(a_i) * f(b_1) * \ldots..) * f(b_m) = 0$. Due to the identity (1. 0) and Lemma 1.1, $f(a_i) * (f(b_1) \vee \ldots.. \vee f(b_m)) = 0$ follows. Thus, $f(a_i) \leq f(b_1) \vee \ldots.. \vee f(b_m)$. Now it is easy to see that $f(a_1) \vee \ldots.. \vee f(a_n) = f(b_1) \vee \ldots.. \vee f(b_m)$, and $\bar{f}$ is, indeed, meaningful.

Of course, $\bar{f}$ preserves the supremum, which is the condition(S) - operation. That $\bar{f}$ preserves $*$, is an easy consequence of (1. 0), 1.2 and 1.3; the calculation is routine. Because $\bar{f}$ is now a BCK-homomorphism, it is one-to-one if and only if its kernel $\{K \in F(A): \bar{f}(K) = 0\}$ is the smallest element of $F(A)$. Keeping this in mind, quickly leads to: $\bar{f}$ is one-to-one when and only when f is. The uniqueness of $\bar{f}$ is not hard.

Let g: $(A; *, 0) \longrightarrow (B; *, 0)$ be a homomorphism between positive implicative BCK-algebras. We avail ourselves of the results and concepts of Category Theory, especially in connection with reflective subcategories; Balbes and Dwinger [1; Chapter 1, especially Section 18 and Theorem 2,

therein]. From 1.4, it thus follows that when $F(g)$ is defined by $F(g)$ $(\langle a_1, \ldots, a_n \rangle) = \langle g(a_1), \ldots, g(a_n) \rangle$, $a_1, \ldots, a_n \in A$, F becomes a functor from the category U into its subcategory V and η is a natural transformation of the identity functor, id U, on U to V, in the notation of Section 0, and it is a reflector and V is a reflective subcategory. We summarize in 1.6, below. However, we first deduce the next result which, at least historically, has independent interest - we comment on this in the proof.

Lemma 1.5. *The variety of* V *of positive implicative* BCK*-algebra with condition*(S) *and its homomorphisms is a full subcategory of the variety* U *of positive implicative* BCK*-algebras.*

Proof. Let $g: A \to B$ be a homomorphism between the positive implicative BCK-algebras $(A; *, 0)$ and $(B; *, 0)$, each of which enjoys the condition(S) - in both cases we employ the supremum-notation for the associated condition(S) - operation. Because of 1.0, 1.1 and 1.3, $\eta_A(a_1 \vee a_2) = \langle a_1 \rangle \vee \langle a_2 \rangle$. This also means that η_A is onto, (c.f. Iséki [4; Corollary 5, p.118]), and so an isomorphism of BCK-algebras with condition(S). As η is a natural transformation, we obtain $g = \eta_B^{-1} F(g) \eta_A$. Because $F(g)$ preserves the (S)-operation, g does also. Thus, V is a full subcategory. Before closing the proof we mention that the version of 1.5 is well known, when g is onto; it is essentially due to Iséki - see the author's paper [3; Lemma 1.9] for details.

Therefore, we obtain:

Theorem 1.6. *The variety* V *of positive implicative* BCK*-algebras with*

condition(S) *and homomorphisms is a full reflective subcategory of the variety* U *of positive implicative* BCK-*algebras and homomorphisms. The reflector,* F: U → V, *is monopreserving and monoreflecting and* η: id U → F *is the front-adjunction.*

Now, let us turn to results which have the flavour of Universal Algebra. We need the following notation. Let $(A; *, 0)$ be a BCK-algebra and $a, b \in A$. Then, the smallest BCK-congruence which identifies a and b is denoted by $\Theta(a, b)$. But we use Θ_b in place of $\Theta(0, b)$.

As U is a variety, it is closed under the formation of homorphic images. Of course, these images are BCK-algebras. Therefore, for any BCK-congruence Φ on a positive implicative algebra A, $a \equiv b(\Phi)$ $(a, b \in A)$ if and only if $a * b, b * a \in \ker \Phi = \{c \in A: c \equiv 0(\Phi)\}$, c.f. Yutani [14; Proposition 3]. In addition, this type of reasoning implies that any BCK-algebra A in a class K of BCK-algebras, which is closed under homomorphic images, is 0-regular, in the sense that the map $\Phi \to \ker \Phi$ is a lattice-isomorphism from the congruence-lattice, Con(A), onto the ideal-lattice, $J(A)$.

Theorem 1.7. *Let* K *be a class of* BCK-*algebras which is closed under the formation of homomorphic images. Then, the following conditions are equivalent.*

(i) *Each algebra in* K *is positive implicative.*

(ii) *Let* $p(x, y, z)$ *be the quaternary polynomial* $p(x, y, z, w)$ *given by* $p(x, y, z, w) = ((x * y) * (z * w)) * (w * z)$. *Then, on any* K-*algebra* A, $c \equiv d(\Theta(a, b))$ *if and only if* $p(c, d, a, b) = 0 = p(d, c, a, b)$

on A, *for any* $a, b, c, d, \in A$.

(iii) *For any* K-*algebra* A, *and any* $b, c, d, \in A$, $c \equiv d(\Theta_b)$ *if and only if* $c * b = d * b$.

(iv) *For any* K-*algebra, and any* $b, c \in A$, $c \in \langle b \rangle$ *if and only if* $c \leq b$.

Proof. (i) $\Rightarrow$ (ii). Because a positive implicative algebra A is 0-regular, it is not hard to see that $\Theta(a, b)$ is the congruence induced by the ideal $\langle a * b, b * a \rangle$. Then, $c \equiv d(\Theta(a, b))$ iff $c * d, d * c \equiv 0(\Theta(a, b))$. And, $c * d, d * c \in \langle a * b, b * a \rangle$ is another equivalent condition. Hence, (ii) derives from part (ii) of Theorem 1.3.

(i) $\Rightarrow$ (iii). In a positive implicative BCK-algebra A, the map $a \to a * b$ is a BCK-endomorphism. Hence, the Θ_b of (iii) is a congruence. As its kernel is $\langle b \rangle$, it must be $\Theta_b = \Theta(0, b)$.

(iii) $\Rightarrow$ (iv) is very easy.

(ii) $\Rightarrow$ (iv) holds because of 0-regularity and the identities $p(z, 0, 0, y) = z * y$ and $p(0, z, 0, y) = 0$.

Thus, it remains to show that (iv) implies (i). Thus, assume that (iv) holds and do not forget that K is closed under homomorphic images. Here, we use the equivalent condition (iii) of 1.2. Thus, let J be an ideal of $A \in \mathsf{K}$ and $a, b, c \in A$ be such that $(b * a) * c, a * c \in J$. Let $\bar{A}$ denote the quotient algebra $A/\Theta(J)$, $\bar{x}$ be the $\Theta(J)$-congruence of $x \in A$, and $\Theta(J)$ be the smallest congruence with J as its kernel ——— $r \equiv s\ \Theta(J)$ $(r, s \in A)$ iff $r * s, s * r \in J$. The assumptions, and notation, say that $(\bar{b} * \bar{a}) * \bar{c} = 0 = \bar{a} * \bar{c}$. Hence, *on* $\bar{A}$, $\bar{b} * \bar{a} \equiv 0 \equiv \bar{a}(\Theta(0, \bar{c}))$. Consequently, $\bar{b} \equiv 0\ \Theta(0, \bar{c})$. In other words, $\bar{b}$ is in the

ideal $\langle\bar{c}\rangle$ of $\bar{A}$. Our hypothesis enables us to assert that $\bar{b} * \bar{c} = 0$. That is, $b * c \in J$. Hence, the arbitrary ideal J is implicative and we have finished.

Thus, 1.3 is very fecund. We could conclude more, but we defer until the next section.

2. Congruence-distributivity. Let $(A; \vee)$ be an upper semilattice, which for the sake of simplicity, and direct relevance to the present paper, has a smallest element, 0. By a *semilattice-ideal* of A, we mean, as is usual, a subset K of A such that (i) K is hereditary, in the sense that $x \in K$, whenever $x \in A$ and $x \leq y \in K$, and (ii) K is closed under $\vee$. Such a semilattice is said to be *distributive*, when $a \leq b \vee c$ in A implies $a = b_1 \vee c_1$ for suitable $b_1 \leq b$ and $c_1 \leq c$ in A. This condition is equivalent to the distributivity of the lattice of semilattice-ideals; as a reference, with an extensive bibliography, see the author's paper [2]. In [12; Hilfssatz 3.2], Schmidt showed that a (dual) relatively pseudocomplemented upper semilattice is distributive. The relevance to the present paper of Schmidt's lemma becomes apparent when, in conjunction with Lemma 1.1, we observe that, in a positive BCK-algebra with condition(S), $(A; *, \vee, 0)$, Iséki [4; Corollary 5, p.118] has shown that the BCK-ideals of $(A; *, 0)$ are the same as the semilattice-ideals of $(A; \vee, 0)$. Using this, the 0-regularity-correspondence between BCK-ideals and BCK-congruences on a positive implicative algebra, as used in Section 1, and the well-known lattice-theoretic fact that the algebraic lattice $J(A)$ is lattice-isomorphic to the lattice of semilattice-ideals of its upper subsemilattice $F(A)$ of finitely generated members, we can

say:

Theorem 2.1. *A positive implicative* BCK-*algebra* $(A; *, 0)$ *is congruence-distributive because its congruence-lattice,* Con A, *is lattice-isomorphic to the distributive lattice of all semilattice-ideals, or equivalently all* BCK-*ideals, of the positive implicative algebra* $(F(A); *, \vee, 0)$, *with condition*(S), *of all finitely generated* BCK-*ideals on* A.

In connection, with the congruence-distributivity (BCK-ideal-distributivity, or semilattice-ideal-distributivity) of a positive implicative algebra with condition(S), we wish to point out that a semilattice-proof can be obtained from Nemitz [10; Theorem 4.1], while a Universal Algebra-style of BCK-algebra proof has been given by the author in [3; Theorem 1.10], and its preliminaries. More information can be obtained from 2.1 and the author's Universal Algebra and quasicommutative BCK-algebra results contained in Sections 1 and 2 of [3]. From 2.1 and [3; Theorem 1.2 and Theorem 2.6] it follows that U is a *3-distributive* variety in the sense of Jónsson [8; Theorem 2.1], in the sense that there are two ternary polynomials $t_1(x, y, z)$ and $t_2(x, y, z)$ having the behaviour described in 2.2, below. We refer to Jónsson [8] and the author's paper [3; Sections 2.3] for a perspective, In 2.2, we utilize a variety which contains U and which was mentioned in Section 0. We do not know whether the containment is strict! The comments on 2-distributivity and 2-permutability in 2.2 are clear from the results, [3; Theorem 3.15, Theorem 3.14], on the subvariety of U, which consists of all implicative BCK-algebras.

Theorem 2.2. *The variety* U *of positive implicative* BCK-*algebras is*

3-distributive because the polyn mials $t_1(x, y, z)$, $t_2(x, y, z)$ *given by*:

$$t_1(x, y, z) = (x * ((x * y) * (z * y))) * ((y * x) * (y * z))$$

$$t_2(x, y, z) = (z * ((y * x) * (y * z))) * ((x * y) * (z * y))$$

satisfy the following identities on the variety, containing U, *of all* BCK-*algebras satisfying* $(x * (x * y)) * (y * x) = (y * (y * x)) * (x * y)$. *The identities for* t_1 *and* t_2 *are*:

$$t_1(x, y, x) = x = t_2(x, y, x);$$

$$t_1(x, x, z) = x; \quad t_2(x, x, z) = z;$$

$$t_1(x, z, z) = t_2(x, z, z).$$

Hence, both varieties are 3-distributive and 3-permutable and yet neither is 2-distributive or 2-permutable.

In conclusion, we would like to say that this paper depends heavily on the support, which Professor Kiyoshi Iséki has given us over the years. Many thanks to him!

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School of Mathematical Sciences
The Flinders University of South Australia
Bedford Park, South Australia
Australia