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Lajos, Sándor

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Improvisations Involving Interior Ideals

By Sándor LAJOS

Let S be a semigroup. A subsemigroup A of S is called an interior ideal of S if the inclusion $SAS \subset A$ holds. The principal interior ideal of S generated by an arbitrary element a of S will be denoted by $I(a)$. It is easy to see that every two-sided ideal of S is an interior ideal, but not conversely.

Let us consider the following example:

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	1
4	0	0	0	0	1	2
5	0	0	1	0	3	0

The subsemigroups $\{0, 3\}$, $\{0, 5\}$, $\{0, 2, 3\}$, $\{0, 1, 4\}$, $\{0, 1, 5\}$ are interior ideals which are not two-sided ideals.

In this short note some characterisations will be given for certain classes of semigroups. For the undefined notions and notations we refer to A. H. Clifford and G. B. Preston [1].

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First we give a characterization of regular semigroups.

Theorem 1. *For a semigroup S the following conditions are equivalent:*

(A) *S is regular.*

(B) *$B \cap I = BIB$ holds for every bi-ideal B and every interior ideal I of S.*

(C) *$B(a) \cap I(b) = B(a)I(b)B(a)$ for every couple a, b in S.*

(D) *$B(a) \cap I(a) = B(a)I(a)B(a)$ for every element a of S.*

Proof. (A) implies (B). Let S be a regular semigroup. Suppose that B is a bi-ideal, I is an interior ideal of S and $a \in B \cap I$. Then there is an element x in S such that $axa = a$. Hence it follows that $a = (axa)(xax)a \in BIB$. Thus we have

$$B \cap I \subset BIB \subset B \cap I.$$

Evidently, (B) implies (C) and (C) implies (D).

Finally, we show that (D) implies (A). Assume that condition (D) holds for a semigroup S. Then it follows that

$$a \in B(a) \cap I(a) = B(a)I(a)B(a) \subset aSa$$

holds for every element a of S. Therefore S is regular, indeed.

Corollary. *An element a of a semigroup S is regular if and only if $B(a) \cap I(a) = B(a)I(a)B(a)$.*

For other characterizations of regular elements, see [3], [4].

Theorem 2. *An element a of a semigroup S is both regular and intraregular if and only if $L(a) \cap R(a) = R(a)I(a^2)L(a)$.*

Proof. First, assume that an element a of a semigroup S is both

regular and intraregular, i.e. there exist elements x, y in S such that $a = axa^2ya$. Let $b \in L(a) \cap R(a)$. Then we have

$$L(a) \cap R(a) = aS \cap Sa = aS^2a,$$

and $b = ata$, where $t \in S^2$. Therefore $b = (axa^2ya)ta = (ax)a(aya)ta$ which belongs to the product $R(a)I(a^2)L(a)$, and we conclude that

$$L(a) \cap R(a) \subset R(a)I(a^2)L(a).$$

The opposite inclusion holds for any element a of S.

Conversely, if the above condition of Theorem 2 holds for an element a of a semigroup S, then we have

$$a \in L(a) \cap R(a) = R(a)I(a^2)L(a) = (aS^1)(a^2 \cup Sa^2S)(S^1a),$$

whence it follows that a is both regular and intraregular.

Corollary. *A semigroup S is both regular and intraregular if and only if $L(a) \cap R(a) = R(a)I(a^2)L(a)$ holds for every element a of S.*

The proof of the following criterions is straightforward.

Theorem 3. *A semigroup S is a semilattice of right groups if and only if the condition $B \cap I = IB$ holds for every bi-ideal B and every interior ideal I of S.*

This result together with some earlier ones gives the next theorem.

Theorem 4. *For a semigroup S the following conditions are equivalent:*

(A) *S is a semilattice of right groups.*

(B) *$B \cap R = RB$ for every bi-ideal B and right ideal R of S.*

(C) *$Q \cap R = RQ$ for every quasi-ideal Q and right ideal R of S.*

(D) *$I \cap Q = IQ$ for every two-sided ideal I and every quasi-ideal Q of S.*

(E) $L \cap R = LRL$ *for every left ideal* L *and right ideal* R *of* S.

(F) *The multiplicative semigroup of all bi-ideals of* S *is a right regular band.*

(G) *The bi-ideal semigroup* $B(S)$ *is a right duo band.*

(H) $B(S)$ *is a band which is a semilattice of right zero semigroups.*

(I) $B \cap I = IB$ *for every bi-ideal* B *and interior ideal* I *of* S.

(J) $B \cap I = IB$ *for every bi-ideal* B *and every two-sided ideal* I *of* S.

(K) $I \cap X = IX$ *for every two-sided ideal* I *and every one-sided (left or right) ideal* X *of* S.

(L) *For every element* a *of* S *there is an idempotent element* $e = e(a)$ *such that* $B(a) = Se$.

(M) *The collection* $12(S)$ *of all* $(1,2)$*-ideals of* S *is a right regular band with respect to set product.*

(N) S *is regular and* $Se \subset eS$ *holds for every idempotent* e *of* S.

Finally we announce the following criterion.

Theorem 5. *A semigroup* S *is completely regular if and only if the condition* $A \cap I = AIA^2$ *holds for every* $(1,2)$*-ideal* A *and every interior ideal* I *of* S.

For further results concerning interior ideals of semigroups, see [9].

References

[1] A. H. Clifford and G. B. Preston: The algebraic theory of semigroups, Vol. I, Amer. Math. Soc., Providence, R. I., 1964.

[2] S. Lajos: (m,k,n)-ideals in semigroups, K. Marx Univ. Economics, Dept. Math., no. 1 (1976), 12-19.

[3] S. Lajos: Characterizations of certain elements in semigroups, Math. Japon. 22 (1977), 21-23.

[4] S. Lajos: Characterizations of certain elements in semigroups, K. Marx Univ. Economics, Dept. Math., no. 3 (1977), 18-22.

[5] S. Lajos: A characterization of Cliffordian semigroups, Proc. Japan Acad., 52 (1976), 496-497.

[6] S. Lajos: On semigroups that are semilattices of left groups, Mat. Vesnik (Beograd), 11 (1974), 125-126.

[7] S. Lajos: (1,2)-ideal characterizations of certain classes of semigroups, these Notes, 7 (1979), 139-142.

[8] S. Lajos: Contributions to the ideal theory of semigroups, Acta Sci. Math., 42 (1980), 117-120.

[9] G. Szász: Interior ideals in semigroups, K. Marx Univ. Economics, Dept. Math., no. 5 (1977), 1-7.

Pincer u. 8. H-1194 Budapest
Hungary