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Prime Links, Concordance and Alexander Invariants

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Bleiler [1] shows that any knot is concordant to a prime knot with the same Alexander polynomial. In this paper, we prove that any link is concordant to a prime link with the same Alexander invariant. To establish this, we use a variant of the method recently used by Kirby and Lickolish to prove the primeness of certain knots [4] and surgical descriptions of links [5]. The reader is referred to Rolfsen's book [6] for the standard results and definitions of link theory and should interpret this paper in the *PL* category.

In this paper, the word "*tangle*" is used to mean a set of two disjoint arcs properly embedded in a 3-ball. Following [4], we define a tangle t in a 3-ball B is *prime* if it has the following properties:

(1) Any 2-sphere in B, which meets t transversely in two points, bounds in B a ball meeting t in an unknotted spanning arc.

(2) The arcs of t cannot be separated by a disc properly embedded in B.

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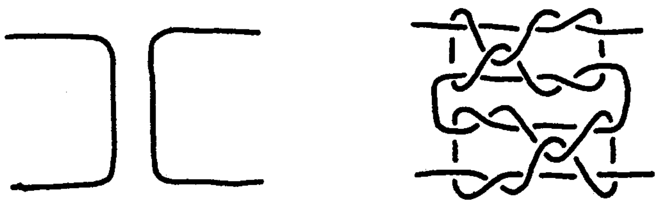

(a) (b)

Fig. 1

In Fig. 1, (a) is not a prime tangle; which is named *a trivial tangle*, and (b) is a prime tangle, which is named *the* $K \overset{d}{-} T$ *grabber*.

Lemma 1. *The* $K \overset{d}{-} T$ *grabber is a prime tangle.*

Proof. Condition (1) is immediate as each arc is unknotted. If condition (2) fails, this tangle with "ears" as in Fig. 2 must be a 2-bridge link. The two fold branched covering space M of a 2-bridge link L is a lens space and the first homology group of M, $H_1(M)$, is isomorphic to $Z_{|\Delta_L(-1)|}$, where $\Delta_L(t)$ is a reduced Alexander polynomial of L. This argument shows that the above link is a trivial link with 2-components. Then a component of the link bounds a disc missing another component. But this is false.

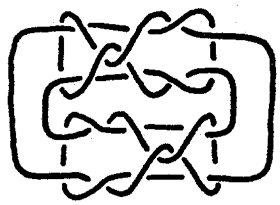

Fig. 2

Lemma 2. *Let* $L' = L \cup k$ *be a link, where* k *is a trivial knot. Suppose that* k *has linking number* 0 *with each component of* L *and that* k *bounds a ribbon disc in* $S^3 - L$. *We denote by* L^* *a link which is obtained from* L *by a surgery along* k *in* S^3. *Then both* L *and*

L^* have the same Alexander invariant.

Proof. We use a variant of surgical descriptions of links ([5]). We denote by $\tilde{X}_a$ and $\tilde{X}_a{}^*$ the universal abelian covering spaces of L and L^* respectively. The above surgery curve k is lifted to $\tilde{X}_a$ and the surgeries along all the lifts of k give us $\tilde{X}_a{}^*$. On the other hand, all the lifts of k bound ribbon discs in $\tilde{X}_a$. Hence these surgeries are homologicaly non-effective. As the Alexander invariant is a homological invariant of the universal abelian covering space, both L and L^* have the same Alexander invariant.

Lemma 3. *The substitution* (b) *for* (a) *on a regular projection of a link does not change the Alexander invariant and the concordance class.*

Proof. From a surgical description of (b) as in Fig. 3x, which is deformed to Fig. 3y, we see these surgery curves bound ribbon discs in the complement of the tangle in the 3-ball. From Lemma 2, this substitution does not change the Alexander invariant. Ribbon moves at β_1 and β_2 in Fig. 3x show that (a) is concordant to (b).

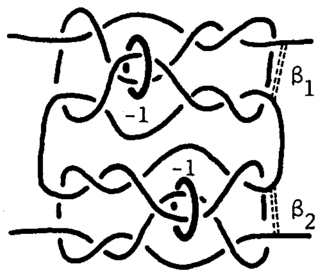

Fig. 3x

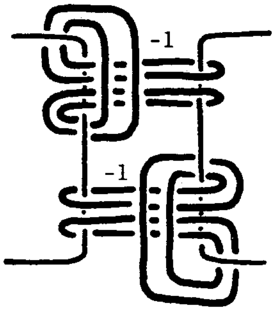

Fig. 3y

A link L is called *separable* if there exists a 2-sphere S such that $S \cap L = \emptyset$ and that each component of $S^3 - S$ contains a component of L.

Lemma 4. *Let L be a 2-components link which is obtained from the $K \overset{d}{-} T$ grabber in a 3-ball B by joining two arcs in the exterior of B. Then L is non-separable.*

Proof. Suppose that L is separable. Then there exists a 2-sphere S such that $S \cap L = \emptyset$ and each component of $S^3 - S$ contains a component of L. We may assume that $S \cap \partial B$ consists of simple closed curves where ∂B means the boundary of B. Let γ be a curve of $S \cap \partial B$, which bounds a disc δ on S and bounds a disc ε on ∂B. $\delta \cup \varepsilon$ is an immersed S^2 in S^3. In general, a simple closed curve and an immersed S^2 in S^3 meet transversely in an even number of points. From this, we see that $\delta \cup \varepsilon$ and a component of L meet in an even number of points. Since $\delta \cap L = \emptyset$, ε and a component of L meet in 0 or 2 points. Hence we can replace the joining arcs by other new joining arcs which are parallel to ∂B and do not meet S. This new link L' is separable, too.

We may draw a regular projection of L' as in Fig. 4, where r is a "*rational*" tangle ([2]). Since L' is separable, $\Delta_{L'}(t) = 0$. We denote by L^* a link which is obtained from L' by a substitution (a) for (b) in Fig. 4. From Lemma 3, we have $\Delta_{L^*}(t) = 0$. From Conway's assertion about rational tangles [2], the tangle r

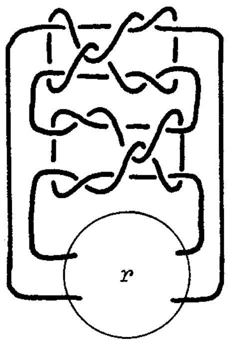

Fig. 4

is a trivial tangle. Hence L' is the link preceding in Proof of Lemma 1. But this contradicts that L' is separable. The proof is complete.

Lemma 5. ([4]) *Let* k *be a prime non-trivial knot in* S^3. *Then there exists, embedded in* S^3, *a 2-sphere meeting* k *transversely in four points and separating* S^3 *into 3-balls* A *and* B *such that*

(i) $(A, A \cap k)$ *is a trivial tangle;*

(ii) $(B, B \cap k)$ *is a prime tangle.*

A link L is called *prime* if there never exists a 2-sphere S such that $S \cap L$ consists of two points and that each component of S^3-S meets L in an knotted spanning arc or in an arc and loops.

Lemma 6. *Let* k *be a prime non-separable 2-components link one of whose components is non-trivial. Then there exists a 2-sphere satisfying the condition of Lemma* 5.

Proof. Let A be a 3-ball in S^3 such that $(A, A \cap k)$ is a trivial tangle and that A contains a subarc of each component of k. Let B be the exterior of A. We will show that $(B, B \cap k)$ is a prime tangle. Condition (1) is immediate as k is prime. $B \cap k$ contains a knotted arc from the assumption. This fact and condition (1) imply condition (2).

Lemma 7. *Let* k *be a 2-components link which is a connected sum of* k_1 *and* k_2, *where* k_1 *is a prime non-trivial knot and* k_2 *is a non-separable 2-components link whose components are trivial. Then there exists a 2-sphere satisfying the condition of Lemma* 5.

Proof. From Lemma 5, we may consider a regular projection of k as in Fig. 5 where t_1 is a prime tangle associated with k_1 and t_2 is a tangle associated with k_2. We show that $t_1 \cup t_2$ in $B_1 \cup B_2$ is a prime tangle. Let S be a 2-sphere in $B_1 \cup B_2$, which meets $t_1 \cup t_2$ transversely in two points. We may assume that $S \cap D$ consists of simple closed curves.

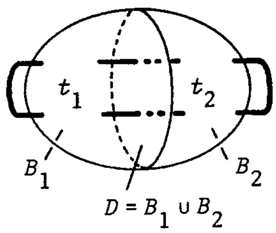

Fig. 5

First we will prove $S \cap D$ can be removed preserving the knot type of $(B', B' \cap k)$ where B' is a 3-ball bounded by S in $B_1 \cup B_2$. We can choose $\gamma \in S \cap D$ and a disc δ on S such that $\partial\delta = \gamma$ and $\delta \cap (S \cap D) = \gamma$ and that $\delta \cap k$ contains at most one point. We denote by ε a disc on D bounded by γ. Then $\delta \cup \varepsilon$ bounds a 3-ball B^* in B_1 or B_2.

If $\varepsilon \cap k = \emptyset$, then γ is removal from $S \cap D$ since $B^* \cap k = \emptyset$.

If $\varepsilon \cap k$ contains two point, then this contradicts the fact that k is non-separable.

If $\varepsilon \cap k$ contains only one point, then $\delta \cap k$ contains only one point, too. From the primeness of t_1 or the unknottedness of arcs of t_2 in compliance with $B^* \subset B_1$ or $B^* \subset B_2$, $B^* \cap k$ must be unknotted in B^*. Hence we can remove γ preserving the knot type of $(B', B' \cap k)$. Hence we can assume $S \cap D = \emptyset$. Then S is contained in B_1 or B_2. From the primeness of t_1 or the unknottedness of arcs of t_2, B' meets $t_1 \cup t_2$ is an unknotted spanning arc. Hence condition (1) is satisfied.

Condition (1) and the fact that $t_1 \cup t_2$ contains a knotted arc imply condition (2). This establishes Lemma 7.

Remark. Using [1, Lemma 1.1], we can also show Lemma 7 without the primeness of k_1.

Lemma 8. ([4]) *The join of three prime tangles as in Fig. 6, where t_1, t_2 and t_3 are prime tangles, is a prime link.*

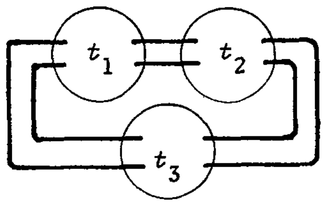

Fig. 6

Proof. See [4, Proof of Theorem].

Theorem. *Any link is concordant to a prime link with the same Alexander invariant.*

Proof. There are three steps. The following deformations never change the Alexander invariant and the concordance class from Lemma 3.

(i) Using a method in [1] and [4] with the $K \overset{d}{-} T$ grabber, we deform each component of a given link to a non-trivial prime one.

(ii) From Lemma 4, we deform each 2-components sublink of a given link to a non-separable one by certain substitutions the $K \overset{d}{-} T$ grabber for a trivial tangle.

(iii) For each non-prime 2-components sublink of a given link, we decompose this sublink into the connected sum of two links, one of which is a non-trivial prime knot and another is a 2-components link satisfying the condition in Lemma 6 or 7.

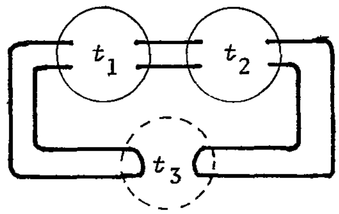

Fig. 7

Using Lemmas 5, 6 and 7, we can consider a regular projection of the sublink as the diagram as in Fig. 7 where both t_1 and t_2 are prime tangles. We substitute the $K \overset{d}{-} T$ grabber for t_3 in Fig. 7 and obtain a new link. From Lemma 8, we know that each 2-components sublink is deformed to a prime (and non-separable) one.

In this way, we obtain a required link because a link, whose each 2-components sublink is non-separable prime, must be prime.

Remark. Deformations in steps (ii) and (iii) do not change a knot type of each component and link types of other 2-components sublinks.

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