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(Citation)
Mathematics Seminar Notes,8(3):573-591

(Issue Date)
1980

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/E0001579

(URL)
https://hdl.handle.net/20.500.14094/E0001579

MATHEMATICS SEMINAR NOTES
Vol. 8 (1980).

Knot-groups of higher dimensional Ribbon Knots

By Takaaki YANAGAWA

We have already proved that *a ribbon n-knot* K^n *in* R^{n+2} *has nice cross-sections which are called equatorial knots* , ref. (2, 8)Theorem in [9] . By making use of the above result and the van Kampen Theorem, we will prove in this paper the following Theorem for ribbon n-knot K^n and its equatorial knot K^{n-1} :

(4, 3)Theorem. *If* $n \geqq 3$, *then* $\pi_1(R^{n+2} - K^n) \cong \pi_1(R^{n+1}(0) - K^{n-1})$.

As the consequences of this theorem, we have two corollaries below.

(4, 4)Corollary. *Assume that* K_α^{n-1} $(\alpha = 1, 2)$ *are two equatorial knots of a ribbon n-knot. If* $n \geqq 3$, *then* $\pi_1(R^{n+1} - K_1^{n-1})$ *is isomorphic to* $\pi_1(R^{n+1} - K_2^{n-1})$.

(4, 9)Corollary. $G(1) \subsetneqq G(2) = G(3) = G(4) = \cdots$, *where* $G(n)$ *denotes the collection of knot-groups* $\pi_1(R^{n+2} - K^n)$ *of ribbon n-knots* K^n .

The author would like to express his much gratitude to Professor K. MURASUGI for many useful conversations which helped him to complete this paper.

This research was partially supported by National Council of CANADA grant No. A4034.
Received December 24, 1980.

1. Notations and Propositions

R^m ; Euclidean m-space.

$R^{m-1}(t)$; an hyperplane in R^m defined by the equation $x_m = t$.

$cl.M$; the closure of the set M in R^m .

$int.M$; the set of interior points of M .

∂M ; the set of boundary points of M .

$M - N$; the set $M - N \cap M$.

Two n-knots *(locally flat n-spheres in R^{n+2})* K_1^n and K_2^n *belongs to the same knot-type,* if there is an orientation preserving homeomorphism $f : R^{n+2} \to R^{n+2}$ such that $f(K_1^n) = K_2^n$. Throughout this paper, we understand everything from the combinatorial standpoint of view. The concepts of *ribbon knots* and *equatorial knots* are due to the following definitions.

(1, 1)Definition. A locally flat n-sphere K^n $(n \geqq 1)$ in R^{n+2} will be called *a ribbon n-knot,* if there is a map ρ from an (n+1)-ball B^{n+1} into R^{n+2} satisfying the following conditions ;

(1, 1, 1) ρ is an immersion and $\rho|\partial B^{n+1}$ is an imbedding such as $\rho(\partial B^{n+1}) = K^n$,

(1, 1, 2) the self-intersection of $\rho(B^{n+1})$ consists of mutually disjoint n-balls $D_1^n, D_2^n, \ldots, D_s^n$, unless ρ is an imbedding,

(1, 1, 3) the inverse set $\rho^{-1}(D_i^n)$ consists of disjoint n-balls D'^n_i and D''^n_i such that $D'^n_i \subset int.B^{n+1}$ and $\partial D''^n_i = D''^n_i \cap \partial B^{n+1}$ ($i = 1, 2, \ldots, s$) .

The map ρ in (1, 1)Def. will be called *a ribbon-map for K^n* , and the more strict definition is given in (2, 1)Def. in [9] .

(1, 2)Definition. A ribbon (n-1)-knot K_0^{n-1} $(n \geqq 2)$ in $R^{n+1}(0)$ will be called *a canonical equatorial knot of a ribbon n-knot* K^n, if there are a ribbon n-knot K^n and a ribbon-map $\rho : B^{n+1} \to R^{n+2}$ for K^n satisfying the following conditions ;

(1, 2, 1) $K_0^{n-1} = K^n \cap R^{n+1}(0)$,

(1, 2, 2) let B^{n+1} be an (n+1)-ball defined by $B^{n+1} = (B_0^n \times [-1,1]) \cup (cl.\{B_0^n - N_1^n \cup N_2^n \cup \ldots \cup N_s^n\} \times [-2, 2])^{(1)}$, where $N_i = h_i(I^{n-1} \times I)$ $(i = 1, 2, \ldots, s)$ are the images of $I^{n-1} \times I$ by h_i associated with a ribbon-map $\rho_0 : B^n \to R^{n+1}(0)$ for K_0^{n-1} . Then, the map $\rho : B^{n+1} \to R^{n+2}$ should be defined by $\rho((x, t)) = (\rho_0(x), t) \in R^{n+1}(t)$.

(1, 3)Definition. A ribbon (n-1)-knot K_α^{n-1} will be called *an equatorial knot of a ribbon n-knot* K^n , if there is a ribbon n-knot K_α^n belonging to the same knot-type as K^n and if K_α^{n-1} is a canonical equatorial knot of K_α^n .

(1, 4)Definition. We will say that a ribbon n-knot K^n *has a canonical equatorial knot* K_0^{n-1} , if a ribbon (n-1)-knot K_0^{n-1} is a canonical equatorial knot of K^n . Also, we will say that a ribbon n-knot K^n *has equatorial knots* K_α^{n-1} , if each ribbon (n-1)-knot K_α^{n-1} is an equatorial knot of K^n .

(1, 5)Proposition. *If* K_0^n $(n \geqq 1)$ *is a ribbon n-knot in* $R^{n+2}(0)$ *, then* K_0^n *is a canonical equatorial knot of a ribbon (n+1)-knot in* R^{n+3}.

(1) $B^{n+1} \subset B_0^n \times [-2, 2]$.

(1, 6)Proposition. *If a ribbon n-knot* K^n *has a canonical equatorial knot, then* K^n *is symmetric with respect to the hyperplane* $R^{n+1}(0)$ *in* R^{n+2} .

(1, 7)Proposition {(2, 8)Theorem in [9]} . *If* $n \geqq 2$, *then any ribbon n-knot has at least one equatorial knot.*

2. Critical band

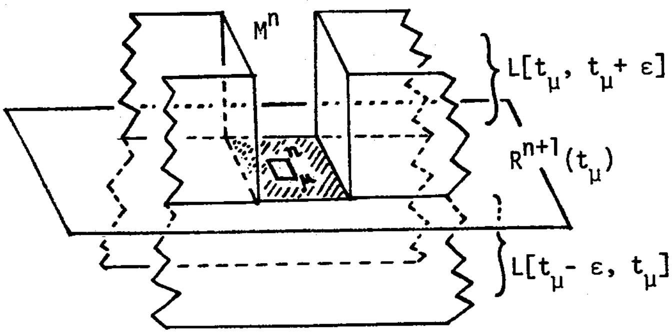
Let M^n be a locally flat, closed (compact, without boundary) orientable n-manifold in R^{n+2} . Denote by $L[\alpha,\beta]$ the region of R^{n+2} defined by the inequality $\alpha \leqq x_{n+2} \leqq \beta$. Let $\pi_\gamma : R^{n+2} \to R^{n+1}(\gamma)$ be the orthogonal projection ; that is, $\pi_\gamma(x_1, x_2,\dots, x_{n+1}, x_{n+2}) = (x_1, x_2,\dots, x_{n+1}, \gamma)$. The notation $\{M^n \cap R^{n+1}(\gamma)\} \boxtimes [\alpha,\beta]$ denotes the set defined by $\{\pi_\gamma^{-1}(M^n \cap R^{n+1}(\gamma)\} \cap L[\alpha,\beta]$ in R^{n+2} , where α,β and γ are real numbers and $\alpha \leqq \beta$. In the case when $\alpha = \beta$, we use the notation $\{M^n \cap R^{n+1}(\gamma)\} \boxtimes [\alpha]$ to denote $\{\pi_\gamma^{-1}(M^n \cap R^{n+1}(\gamma)\} \cap R^{n+1}(\alpha)$.

(2, 1)Definition. We will call M^n *almost everywhere perpendicular* (abbrivated to *a.e.p.*) in R^{n+2} , if there exist a positive number ε , a series of real numbers $t_1, t_2,\dots, t_m$ $(t_1 < t_2 < \dots < t_m)$ and flat n-balls $\square_\mu^n$ (called *critical bands*) in $R^{n+1}(t_\mu)$ $(\mu = 1, 2,\dots, m)$ satisfying the conditions (2, 1, 1) to (2, 1, 3) :

(2, 1, 1).

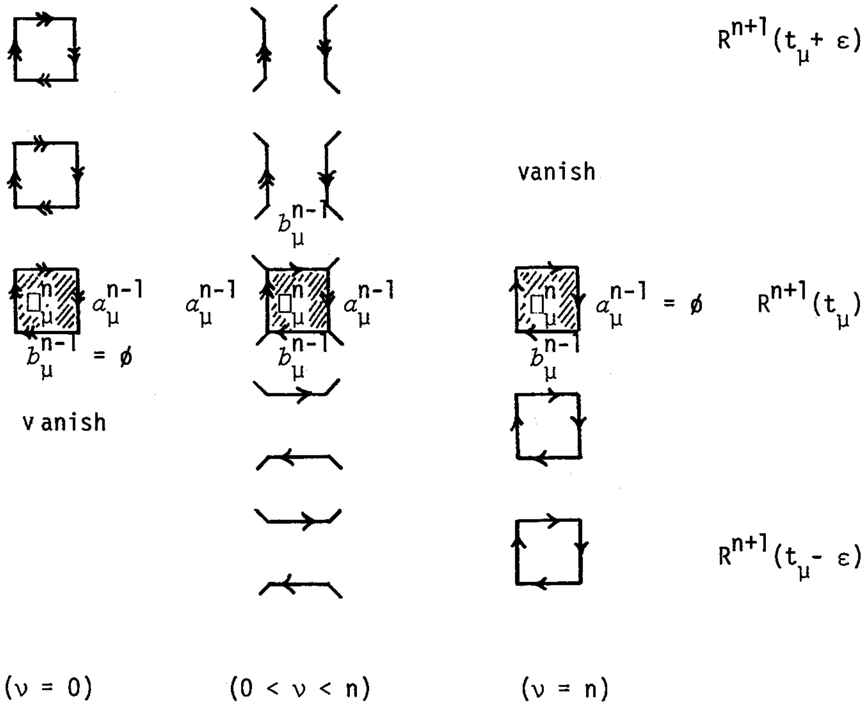
$$M^n \cap L[t_\mu, t_\mu+ \varepsilon] = (\{M^n \cap R^{n+1}(t_\mu+ \varepsilon)\} \boxtimes [t_\mu, t_\mu +\varepsilon]) \cup \square_\mu^n$$
$$M^n \cap L[t_\mu- \varepsilon, t_\mu] = (\{M^n \cap R^{n+1}(t_\mu- \varepsilon)\} \boxtimes [t_\mu- \varepsilon, t_\mu]) \cup \square_\mu^n$$

for each μ $(\mu = 1, 2,\dots, m)$, see Fig.1 .



M^n
$L[t_\mu, t_\mu + \varepsilon]$
$R^{n+1}(t_\mu)$
$\square^n_\mu$
$L[t_\mu - \varepsilon, t_\mu]$

Fig.1



$R^{n+1}(t_\mu + \varepsilon)$
vanish
b^{n-1}_μ
a^{n-1}_μ
$b^{n-1}_\mu = \phi$
$a^{n-1}_\mu = \phi$
$R^{n+1}(t_\mu)$
vanish
$R^{n+1}(t_\mu - \varepsilon)$
$(\nu = 0)$
$(0 < \nu < n)$
$(\nu = n)$

Fig.2

(2, 1, 2).

$(\{M^n \cap R^{n+1}(t_\mu + \varepsilon)\} \boxtimes [t_\mu]) \cap \square^n_\mu = a^{n-1}_\mu \subset \partial\square^n_\mu$

$(\{M^n \cap R^{n+1}(t_\mu - \varepsilon)\} \boxtimes [t_\mu]) \cap \square^n_\mu = b^{n-1}_\mu \subset \partial\square^n_\mu$, and there exists a homeomorphism $h_\mu : \square^n_\mu \to I^{n-\nu} \times I^\nu$ such that $h_\mu(a^{n-1}_\mu) = \partial I^{n-\nu} \times I^\nu$ and $h_\mu(b^{n-1}_\mu) = I^{n-\nu} \times \partial I^\nu$ ($\mu = 1, 2, \ldots, m$), where the integer ν ($0 \leqq \nu \leqq n$) will be called *the index of* $\square^n_\mu$, see Fig.2 .

(2, 1, 3). Whenever $t \neq t_\mu$ ($\mu = 1, 2, \ldots, m$), $M^n \cap R^{n+1}(t)$ consists of a finite number of locally flat, closed (n-1)-manifolds in $R^{n+1}(t)$ (2).

(2, 2)Definition. We will call the n-ball $\square^n_\nu$ in (2, 1) *a critical band of index* ν .

(2, 3)Definition. We will call the level $R^{n+1}(t_\mu)$ *an exceptional level of index* ν and the level $R^{n+1}(t)$ ($t \neq t_\mu$) *an ordinal level* ($\mu = 1, 2, \ldots, m$) .

(2, 4)Theorem. *Assume that* K^{n-1} *is an equatorial knot of a ribbon n-knot* K^n . *If* $n \geqq 2$ *, then there is a ribbon n-knot* $\tilde{K}^n$ *satisfying the conditions (2, 4, 1) to (2, 4, 4) below :*

(2, 4, 1). $\tilde{K}^n$ *belongs to the same knot-type as* K^n .

(2, 4, 2). $\tilde{K}^n \cap R^{n+1}(0)$ *belongs to the same knot-type as* K^{n-1} .

(2, 4, 3). $\tilde{K}^n$ *is a.e.p. in* R^{n+2} *and the indices of the critical levels above* $R^{n+1}(0)$ *are either (n-1) or n .*

(2, 4, 4). $\tilde{K}^n$ *is symmetric with respect to* $R^{n+1}(0)$.

Proof. By the assumption that K^{n-1} is an equatorial knot of given

(2) If $M^n \cap R^{n+1}(t) \neq \emptyset$.

ribbon n-knot K^n , there is a ribbon n-knot K^n_* , a ribbon-map ρ_* for K^n_* and a ribbon-map $\rho_0 : B^n_0 \to K^{n-1}$ for K^{n-1} which satisfy conditions for K^n , ρ and ρ_0 for K^{n-1}_0 in (1, 2)Def.. Let $h_i : I^{n-1} \times I \to B^n_0$ be the imbeddings in (2, 3, 2) in [9] . Move $\rho_0 h_i(I^{n-1} \times [\alpha_i, \beta_i])$ slightly by an orientation preserving homeomorphism $\theta : R^{n+1}(0) \to R^{n+1}(0)$ so that $\theta\rho_0 h_i(I^{n-1} \times [\alpha_i, \beta_i])$ are contained in the hyperplane defined by $x_{n+1} = x_{n+2} = 0$ ($i = 1, 2,\dots, s$ and $-1 < \alpha_i < \beta_i < 1$, $t_i \in [\alpha_i, \beta_i]$). Define a map $\Theta : R^{n+2} \to R^{n+2}$ by $\Theta((X, x_{n+2})) = (\theta(X), x_{n+2})$. We can find an orientation preserving homeomorphism $\Lambda : R^{n+2} \to R^{n+2}$ such that $\Lambda | R^{n+1}(0) = \text{id.}$, $\pi_0 \Lambda = \pi_0$ and $\Lambda\Theta(K^n_*)$ is *a.e.p.* in R^{n+2} . Let us define $\tilde{K}^n$ by $\tilde{K}^n = \Lambda\Theta(K^n_*)$. Now, we have that the critical bands of $\tilde{K}^n$ consists of $\{\square_i'\} \boxtimes [i - 1/4]$, $\{\square_i'\} \boxtimes [-i + 1/4]$, and $\{\rho_0(A^n_j)\} \boxtimes [2s + j]$ and $\{\rho_0(A^n)\} \boxtimes [-2s - j]$ ($i = 1, 2,\dots, s$ and $j = 0, 1, 2,\dots, s$) (3) , where $\square_i' = \theta\rho_0 h\,(I^{n-1} \times [\alpha_i, \beta_i])$ and A^n_j are n-balls of $cl.\{B^n_0 - N_1 \cup N_2 \cup \dots \cup N_s\}$ for $N_i = h_i(I^{n-1} \times I)$ ($i = 1, 2,\dots, s$) . Thus the proof is completed.

(2, 5)Remark. The indices of the exceptional levels for K^n are as below ; for $i = 1, 2,\dots, s$ and $j = 0, 1, 2,\dots, s$,

the index of the exceptional level $R^{n+1}(-2s - j)$ is 0 ,

the index of the exceptional level $R^{n+1}(-i + 1/4)$ is 1 ,

the index of the exceptional level $R^{n+1}(i - 1/4)$ is (n-1) and

the index of the exceptional level $R^{n+1}(2s + j)$ is n .

---

(3) We may suppose that $\rho_0(A^n_j)$ are flat n-balls in $R^{n+1}(0)$.

3. Application of the van Kampen Theorem

Let M^n be a locally flat, closed n-manifold which is *a.e.p.* in R^{n+2}. Let $R^{n+1}(0)$ be an exceptional level with index ν . Without loss of generality, we may suppose that the critical band $\square^n$ at the exceptional level $R^{n+1}(0)$ can be described by the relations below ;

$$\square^n : \ |x_k| \leqq 1 \quad (k = 1, 2, \dots, n),\ x_{n+1} = x_{n+2} = 0 .$$

Moreover, we may assume that, in a sufficiently small cubical *nbh.* U^{n+2} of $\square^n$ in R^{n+2} , M^n is made of the blocks of the hyperplanes which are parallel to the coordinate planes in R^{n+2} . Throughout this section, we require that $n \geqq 3$.

(3, 1)Lemma. *Assume that* $X = X_1 \cup X_2$ *is a separable, regular topological space,* $X_0 = X_1 \cap X_2$ *is a closed, arcwise-connected subspace of* X *and that* $X_i - X_0$ $(i = 1, 2)$ *is an open, arcwise-connected subset of* X . *Assume that each* X_i $(i = 0, 1, 2)$ *is locally 0-, 1- and 2-connected in* X . *Then, the van Kampen Theorem is applicable to the groups* $\pi_1(X_i)$ *and* $\pi_1(X)$ $(i = 0, 1, 2)$.

Proof. Ref. (5, 7) p.667 in [7] .

(3, 2, 0)Lemma. *Assume that the index of* $R^{n+1}(0)$ *is* 0 . *Then ,* $\pi_1(L[-\alpha, \delta] - M^n) \cong Z * \pi_1(L[-\alpha, -\delta] - M^n)$ [4] *for some positive number* δ *such as* $\delta < \alpha$.

Proof. Since M^n is *a.e.p.* in R^{n+2} , there is a positive number ε

(4) Z = *an infinite cyclic group.* $Z * H$ = *a free product of two groups* Z *and* H. $G \cong H$ means *two groups* G *and* H *are isomorphic.*

satisfying (2, 1, 1) and (2, 1, 2) . Let C^{n+2} be the cube defined by

$$C^{n+2}: \ |x_i| \leqq 1+\rho \ \ (i = 1, 2, \ldots, n), \ \ |x_{n+1}| \leqq \rho , \ \ |x_{n+2}| \leqq \varepsilon$$

for a sufficiently small positive number ρ .

Since the index of $R^{n+1}(0)$ is 0 , $(\{M^n \cap R^{n+1}(-\varepsilon)\} \times [0]) \cap \square^n = b^{n-1} = h(I^{n-0} \times \partial I^0) = \phi$ by (2, 1, 2) . That is, $\square^n$ is an isolated component of $M^n \cap R^{n+1}(0)$. Thus we have $M^n \cap C^{n+2} = \square^n \cup (\{\partial \square^n\} \boxtimes [0, \varepsilon])$ and $M^n \cap \partial C^{n+2} = \partial\square^n \boxtimes [\varepsilon]$, see Fig.3 below.

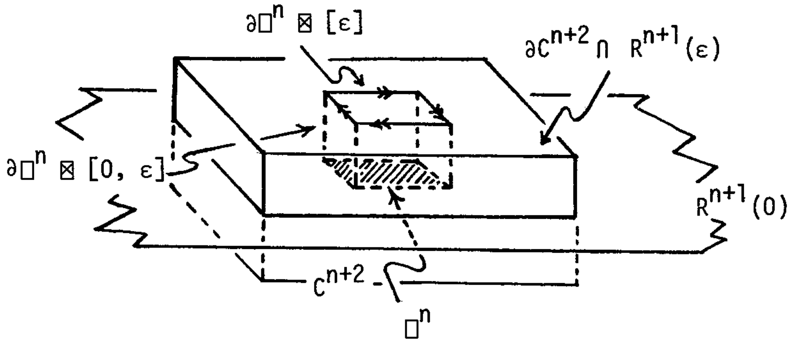


Fig.3

Denote $C^{n+2} - M^n$ by X_1 . Denote $cl.\{L[-\alpha, \varepsilon] - C^{n+2}\} - M^n$ by X_2. Let $X_0 = X_1 \cap X_2$ and $X = X_1 \cup X_2$. Then, X and X_i $(i = 0, 1, 2)$ are arcwise-connected. $X = L[-\alpha, \varepsilon] - M^n$ and $X_0 = cl\{\partial C^{n+2} - R^{n+1}(\varepsilon)\}$. These sets X and X_i $(i = 0, 1, 2)$ satisfy the conditions in (3, 1) . Therefore, by the van Kampen Theorem,

$$\pi_1(X) \cong \pi_1(X_1) * \pi_1(X_2) \cdots\cdots \quad (1)$$

because $\pi_1(x_0) = (1)$. Since $M^n \cap C^{n+2}$ is an unknotted ball[5] in C^{n+2},

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(5) $M^n \cap C^{n+2}$ bounds $\{\square^n\} \boxtimes [0, \varepsilon]$ in C^{n+2} .

we have that $\pi_1(X_1) \cong Z$ ·····(2) . Since M^n is *a.e.p.* in R^{n+2} , there is a deformation retraction $r : X_2 \to L[-\alpha, -\varepsilon] - M^n$ induced by the orthogonal projection $\pi : L[-\varepsilon, \varepsilon] \to R^{n+1}(-\varepsilon)$. Therefore, the map $r_* : \pi_1(L[-\alpha, -\varepsilon] - M^n) \to \pi_1(X_2)$ is an isomorphism. By (1), (2) and the isomorphism r_* above, we have that $\pi_1(X) \cong Z * \pi_1(L[-\alpha, -\varepsilon] - M^n)$. Suppose that $\delta = \varepsilon$, then the proof is completed.

Remark. *$\omega_* : \pi_1(X_2) \to \pi_1(X)$ is an isomorphism, where $\omega : X_2 \to X$ is the inclusion map. Ref. (3, 1) p.63 in 2 . Therefore, the map $\omega_* r_* : \pi_1(L[-\alpha, -\varepsilon] - M^n) \to \pi_1(L[-\alpha, \varepsilon] - M^n)$ is an injection.*

(3, 2, 1)Lemma. *Assume that the index of $R^{n+1}(0)$ is 1. Then, the inclusion map i induces a surjection*

$i_* : \pi_1(L[-\alpha, -\delta] - M^n) \to \pi_1(L[-\alpha, \varepsilon] - M^n)$

for a positive number δ such as $\delta < \varepsilon$.

Proof. Let C^{n+2} be the same cube as in the proof of (3, 2, 0) . Since the index of $R^{n+1}(0)$ is 1 , $(\{M^n \cap R^{n+1}(-\varepsilon)\} \boxtimes [0]) \cap \square^n = b^{n-1} = h(I^{n-1} \times \partial I)$ and $(\{M^n \cap R^{n+1}(\varepsilon)\} \boxtimes [0]) \cap \square^n = a^{n-1} = h(\partial I^{n-1} \times I) = S^{n-2} \times I$ by (2, 1, 2). Since we assume in this section that $n \geqq 3$, $S^{n-2} \times I$ is connected, see Fig.4 . Denote $C^{n+2} - M^n$ by X_1. Denote $cl.\{L[-\alpha, \varepsilon] - C^{n+2}\} - M^n$ by X_2. Let $X_0 = X_1 \cap X_2$ and $X = X_1 \cup X_2$. Then, $X = L[-\alpha, \varepsilon] - M^n$ and $X_0 = cl.\{\partial C^{n+2} - R^{n+1}(\varepsilon)\} - M^n$. In the present case, $M^n \cap cl.\{\partial C^{n+2} - R^{n+1}(\varepsilon)\}$ consists of two (n-1)-balls, $(S_i^{n-1} \boxtimes [-\varepsilon, \varepsilon]) \cup D_i^{n-1}$ $(i = 1, 2)$ and these two (n-1)-balls bounds mutually disjoint n-balls in $cl.\{\partial C^{n+2} - R^{n+1}(\varepsilon)\}$, see Fig.4 again. Therefore, we have that $\pi_1(X_0) = |s_1, s_2 : \text{——} \ |$ ·····(1) , where s_i

is represented by the simple closed curves γ_i $(i = 1, 2)$ which are contained in $\partial\{C^{n+2} \cap R^{n+1}(-\varepsilon)\} - S_1^{n-1} \cup S_2^{n-1}$, see Fig.5 .

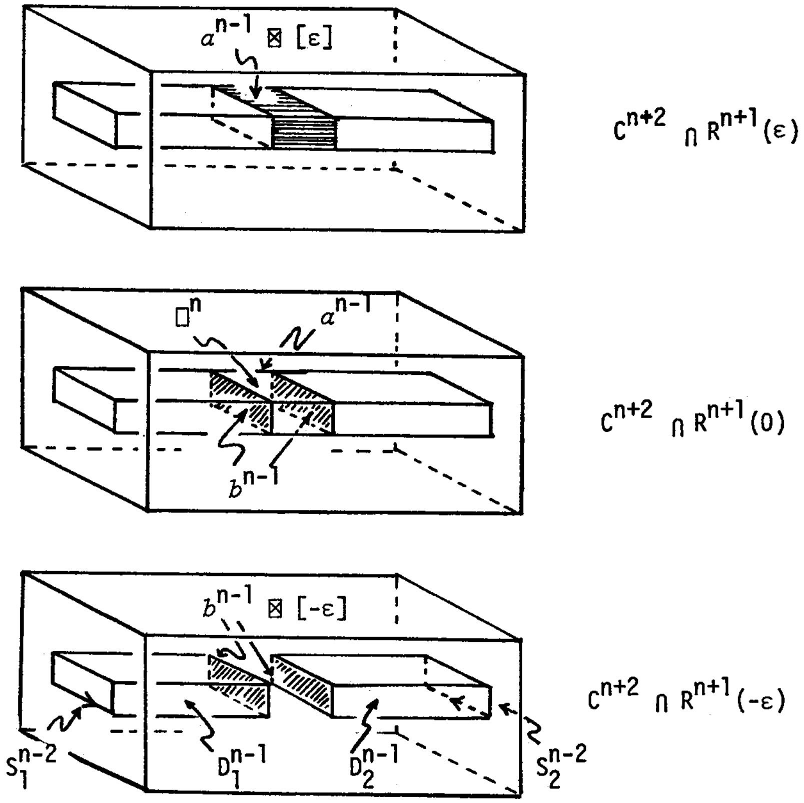

Fig.4

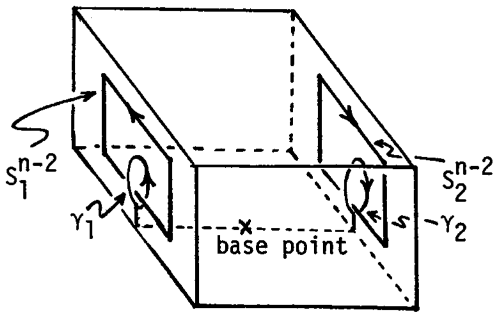

Fig.5

Now, $\pi_1(X_1) = |x_1, x_2 : x_1x_2^{-1} = 1| \cdots\cdots (2)$, where the inclusion $X_0 \subset X_1$ induces a map θ_1 such that $\theta_1(s_i) = x_i$ $(i = 1, 2)$. Assume that $\pi_2(X_2) = |y_1, y_2, y_3, \ldots, y_\mu : r = 1| \cdots\cdots (3)$, where the inclusion $X_0 \subset X_2$ induces a map θ_2 such that $\theta_2(s_i) = y_i$ $(i = 1, 2)$. By applying the van Kampen Theorem to the groups in (1), (2) and (3) , we have that

$$\pi_1(X) = \left| \begin{matrix} x_1, x_2 \\ y_1, y_2, y_3, \ldots, y_\mu \end{matrix} \ : \ \begin{matrix} x_1x_2^{-1}, x_1y_1^{-1}, x_2y_2^{-1} \\ r \end{matrix} \right|_\phi ,$$

ref. (3, 6)Alternative Formulation of The van Kampen Theorem, p.71 in [2].

Then,

$$\pi_1(X) = | \ y_1, y_2, y_3, \ldots, y_\mu \ : \ r , \ y_1y_2^{-1} \ |_\phi$$

$$= | \ v_1, v_2, v_3, \ldots, v_\mu \ : \ \omega_*(r) = 1, \ v_1v_2^{-1} = 1 \ | \quad , \text{ where}$$

the inclusion $X_2 \subset X$ induces a map ω_* such that $\omega_*(y_i) = v_i$ $(i = 1, 2, \ldots, \mu)$. Obviously, $\omega_* : \pi_1(X_2) \to \pi_1(X)$ is a surjection $\cdots\cdots (4)$.

Since M^n is *a.e.p.* in R^{n+2} , there is a deformation retraction $r : X_2 \to L[-\alpha, -\varepsilon] - M^n$, and we have that

$r_* : \pi_1(L[-\alpha, -\varepsilon] - M^n) \to \pi_1(X_2)$ is an isomorphism $\cdots\cdots (5)$.

By composing two maps in (4) and (5), we have that

$\omega_*r_* : \pi_1([-\alpha, -\varepsilon] - M^n) \to \pi_1(X)$ is a surjection and the proof is completed by taking $\delta = \varepsilon$.

(3, 2, n-1)Lemma. *Assume that the index of* $R^{n+1}(0)$ *is* $(n-1)$. *Then, the inclusion map* i *induces an isomorphism*

$i_* : \pi_1(L[-\alpha, -\delta] - M^n) \to \pi_1(L[-\alpha, \delta] - M^n)$

for a positive number δ *such as* $\delta < \alpha$.

Proof. Let C^{n+2} be as before. Since the index of $R^{n+1}(0)$ is $(n-1)$, it follows that $(\{M^n \cap R^{n+1}(-\varepsilon)\} \boxtimes [0]) \cap \square^n = b^{n-1} = h(I \times \partial I^{n-1}) = I \times S^{n-2}$ and $(\{M^n \cap R^{n+1}(\varepsilon)\} \boxtimes [0]) \cap \square^n = a^{n-1} = h(\partial I \times I^{n-1})$ by (2, 1, 2). That is, the situation of M^n in C^{n+2} is the reverse of that in (3, 2, 1), compare Fig.4 and Fig.6 .

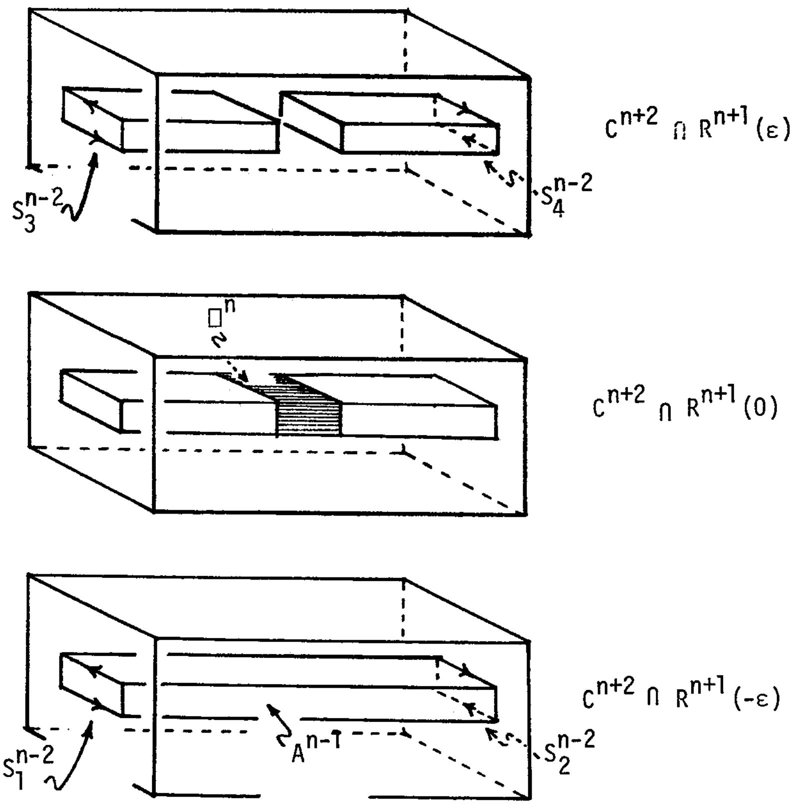

Fig.6

Denote $C^{n+2} - M^n$ by X_1. Denote $cl.\{L[-\alpha, \varepsilon] - C^{n+2}\} - M^n$ by X_2.

Let $X_0 = X_1 \cap X_2$ and $X = X_1 \cup X_2$. Then, $X = L[-\alpha, \varepsilon] - M^n$ and $X_0 = cl.\{\partial C^{n+2} - R^{n+1}(\varepsilon)\} - M^n$. Then, $M^n \cap cl.\{\partial C^{n+2} - R^{n+1}(\varepsilon)\}$ is an (n-1)-ball V^{n-1} which is $\{S_1^{n-2}\} \boxtimes [-\varepsilon, \varepsilon] \cup A^{n-1} \cup \{S_2^{n-2}\} \boxtimes [-\varepsilon, \varepsilon]$. (n-2)-spheres S_3^{n-2} and S_4^{n-2} are contained in $R^{n+1}(\varepsilon)$ and the annulus A^{n-1} is contained in $R^{n+1}(-\varepsilon)$, see Fig.6.

$\pi_1(X_0) = |s_1, s_2 : s_1 s_2^{-1} = 1|$ ····· (1), where s_i are represented by the simple closed curves γ_i ($i = 1, 2$) shown in Fig.5 already. And, the relation $s_1 s_2^{-1} = 1$ is a consequence of the unknottedness of A^{n-1}, that is $\pi_1(\partial C^{n+2} \cap R^{n+1}(-\varepsilon) - A^{n-1}) = Z$. Furthermore,

$\pi_1(X_1) = |x_1, x_2 : x_1 x_2^{-1} = 1|$ ····· (2), where the inclusion $X_0 \subset X_1$ induces the map θ_1 such that $\theta_1(s_i) = x_i$ ($i = 1, 2$). Assume that $\pi_1(X_2) = |y_1, y_2, y_3, \ldots, y_\mu : r = 1|$ ····· (3), where the inclusion $X_0 \subset X_2$ induces the map θ_2 such that $\theta_2(s_i) = y_i$ ($i = 1,2$). Since $X_2 \supset \partial C^{n+2} \cap R^{n+1}(-\varepsilon) - A^{n-1}$, it follows that $y_1 y_2^{-1}$ belongs to the set r. By applying the van Kampen Theorem to the groups in (1) (2) and (3), we have that

$$\pi_1(X) = \left| \begin{matrix} x_1, x_2 \\ y_1, y_2, y_3, \ldots, y_\mu \end{matrix} : \begin{matrix} x_1 x_2^{-1}, x_1 y_1^{-1}, x_2 y_2^{-1} \\ r \end{matrix} \right|_\phi .$$

Then, we have that

$$\begin{aligned} \pi_1(X) &= |y_1, y_2, y_3, \ldots, y_\mu : r, y_1 y_2^{-1}|_\phi \\ &= |y_1, y_2, y_3, \ldots, y : r|_\phi \qquad \text{, because } y_1 y_2^{-1} \in r , \end{aligned}$$

and we have that

$\pi_1(X) = |v_1, v_2, v_3, \ldots, v_\mu : \omega_*(r) = 1|$, where the inclusion $X_2 \subset X$ induces the map ω_* such that $\omega_*(y_i) = v_i$ ($i = 1, 2, \ldots, \mu$). The map $\omega_* : \pi_1(X_2) \to \pi_1(X)$ is an isomorphism ····· (4). Since M^n is *a.e.p.* in R^{n+2}, we have that $\omega_* r_* : \pi_1(L[-\alpha, -\varepsilon] - M^n) \to \pi_1(X)$ is an isomorphism. Suppose that $\delta = \varepsilon$, then the proof is completed.

Remark. *In case of* $n = 2$ *, we obtain only that the map* i_* *mentioned in* (3, 2, n-1) *is surjective. That is,* (3, 2, n-1) *is reduced to the statement in* (3, 2, 1).

(3, 2, n)Lemma. *Assume that the index of* $R^{n+1}(0)$ *is* n *. Then, the inclusion map* i *induces an isomorphism*

$$i_* : \pi_1(L[-\alpha, -\delta] - M^n) \to \pi_1(L[-\alpha, \delta] - M^n)$$

for a positive number δ *such as* $\delta < \varepsilon$.

Proof. The proof is almost clear.

4. G(n)

(4, 1)Definition. Denote the collection of the isomorphism classes of the knot-groups of ribbon n-knot by $G(n)$ $(n = 1, 2, 3, \ldots)$. Let us assume that $\tilde{K}^n$ is a ribbon n-knot which is stated in (2, 4)Theorem. Suppose that the exceptional levels for $\tilde{K}^n$ are as shown in (2, 5)Remark. If $n \geqq 3$, we can prove the existence of the following isomorphisms u , v_i *and* w_i $(i = 1, 2, \ldots, s)$ induced by inclusions ;

(4, 2, 0). $u : \pi_1(R^{n+1}(0) - \tilde{K}^n) \to \pi_1(L[0, 1/2] - \tilde{K}^n)$,

(4, 2, 1). $v_i : \pi_1(L[0, i] - \tilde{K}^n) \to \pi_1(L[0, i + 1/2] - \tilde{K}^n)$ and

(4, 2, 2). $w_i : \pi_1(L[0, i - 1/2] - \tilde{K}^n) \to \pi_1(L[0, i] - \tilde{K}^n)$

$(i = 1, 2, \ldots, s)$.

To prove the existence of the maps in (4, 2, 0) and (4, 2, 1), it is sufficient to prove the existences of deformation retracts r_0 , r_i and $r_i(h)$ such that

$r_0 \quad : L[0, 1/2] - \tilde{K}^n \to R^{n+1}(0) - \tilde{K}^n$,

$r_i \quad : L[i + 1/4, i + 1/2] - \tilde{K}^n \to R^{n+1}(i + 1/4) - \tilde{K}^n$ and

$r_i(h) : L[i, i + 1/4] - \tilde{K}^n \to R^{n+1}(i) - \tilde{K}^n \qquad (i = 1, 2, \ldots, s)$.

r_0 and r_i are given by the orthogonal projection $(i = 1, 2, \ldots, s)$. $r_i(h)$ is realized along the i-th handle $\Lambda\Theta\rho_*(N_i \times [-1, 1])$ for $\tilde{K}^n$, because there is no exceptional level between $R^{n+1}(i)$ and $R^{n+1}(i + 1/4)$ $(i = 1, 2, \ldots, s)$. Since the indices of the exceptional level $R^{n+1}(i - 1/4)$ in $[i - 1/2, i]$ are (n-1) and we assume that $n \geqq 3$ $(i = 1, 2, \ldots, s)$, (4, 2, 2) is a consequence of (3, 2, n-1). Since the indices of the exceptional levels $R^{n+1}(2s + j)$ are n $(j = 0, 1, 2, \ldots, s)$, as a consequence of (3, 2, n), we have the following isomorphisms u_j induced by the inclusion ;

(4, 2, 3)

$$u_j : \pi_1(L[0, 2s + j - 1/2] - \tilde{K}^n) \to \pi_1(L[0, 2s + j + 1/2] - \tilde{K}^n)$$

$(j = 0, 1, 2, \ldots, s)$.

Obviously, we have an isomorphism v induced by inclusion ;

(4, 2, 4)

$$v : \pi_1(L[0, s + 1/2] - \tilde{K}^n) \to \pi_1(L[0, 2s - 1/2] - \tilde{K}^n),$$

because there is no exceptional level between $R^{n+1}(s + 1/2)$ and $R^{n+1}(2s - 1/2)$. Now, we have that

(4, 3) Theorem. *If $n \geqq 3$, $\pi_1(R^{n+2} - \tilde{K}^n) \cong \pi_1(R^{n+1}(0) - \tilde{K}^n \cap R^{n+1}(0))$.*

Proof. Denote by R^{n+2}_+ the upper half-space of R^{n+2}, and by R^{n+1}_- the lower half-space. By means of (4, 2, 0-1-2-3-4), the inclusion induces an isomorphism ω by $\omega = u_s \cdots u_2 u_1 u_0 v v_s w_s \cdots v_1 w_1 u$ as

ω : $\pi_1(R^{n+1}(0) - \tilde{K}^n \cap R^{n+1}(0)) \to \pi_1(L[0, 3s + 1/2] - \tilde{K}^n)$.

Since $\tilde{K}^n \subset L[0, 3s + 1/2]$, we have that

i_* : $\pi_1(R^{n+1}(0) - \tilde{K}^n \cap R^{n+1}(0)) \to \pi_1(R_+^{n+2} - \tilde{K}^n \cap R_+^{n+2})$

is an isomorphism. By the symmetry of K^n , ref. (2, 4, 4), we can easily complete the proof of (4, 3) by applying the van Kampen Theorem to $R_+^{n+2} - \tilde{K}^n$ and $R_-^{n+2} - \tilde{K}^n$.

(4, 4)Corollary. *Assume that* K_α^{n-1} $(\alpha = 1, 2)$ *are two equatorial knots of a ribbon n-knot* K^n. *If* $n \geqq 3$, *then* $\pi_1(R^{n+1} - K_1^{n-1}) \cong \pi_1(R^{n+1} - K_2^{n-1})$.

Proof. Consider $\tilde{K}^n$ for K_α^{n-1} $(\alpha = 1, 2)$. $\tilde{K}_\alpha^n$ belongs to the same knot-type as K^n $(\alpha = 1, 2)$. Then, for $\alpha = 1, 2$, $\pi_1(R^{n+1} - \tilde{K}_\alpha^{n-1}) \cong \pi_1(R^{n+1} - K_\alpha^{n-1})$ by (4, 3), and so $\pi_1(R^{n+1} - K_1^{n-1}) \cong \pi_1(R^{n+1} - K_2^{n-1})$.

Question. *It is an open question whether or not* K_α^{n-1} $(\alpha = 1, 2)$ *belong to different knot-type.*

(4, 5)Lemma. *If* $n \geqq 3$, *then* $G(n) \subseteqq G(n-1)$.

Proof. Let G be a group representing an element of $G(n)$. There is a ribbon n-knot K^n such that $G \cong \pi_1(R^{n+2} - K^n)$. By (1, 7), K^n has an equatorial knot $\tilde{K}^n \cap R^{n+1}(0)$. By (2, 4)Theorem and (4, 3)Theorem this ribbon (n-1)-knot $\tilde{K}^n \cap R^{n+1}(0)$ in $R^{n+1}(0)$ satisfies that $\pi_1(R^{n+1}(0) - \tilde{K}^n \cap R^{n+1}(0)) \cong G$. Then, $G \in G(n-1)$ and the proof is completed.

(4, 6)Lemma. *If* $n \geqq 3$, *then* $G(n-1) \subseteqq G(n)$.

Proof. By (2, 5) in [9], any ribbon (n-1)-knot can be a canonical equatorial knot of a ribbon n-knot. Then, by (2, 4)Theorem and (4, 3) Theorem, the proof is easily seen.

(4, 7)Lemma. *A spun knot $S^2(k)$ in R^4 is a ribbon 2-knot.*

Proof. The image of a ribbon-map $\rho : B^3 \to R^4$ can be given by the rotation of a singular 2-ball $f(D^2)$ bounded by the proper arc k' in R^3_+, see Fig.7.

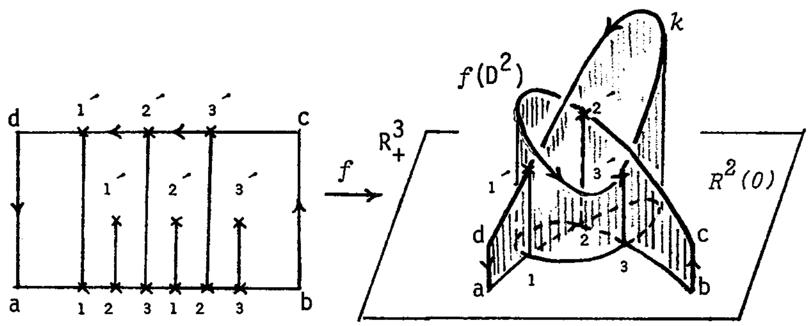

Fig.7

(4, 8)Corollary. $G(1) \subsetneqq G(2)$.

Proof. By (4, 7) and Theorem 3, p.567 in [1], $G(1) \subseteq G(2)$. The following example, Example 10, p.135 in [3], shows that $G(1) \neq G(2)$, also ref. [5] and [8].

(4, 9)Corollary. $G(1) \subsetneqq G(2) = G(3) = G(4) = \cdots\cdots$.

Proof. This is a direct consequence of (4, 5), (4, 6) and (4, 8), also ref. [4] and [6].

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