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A Note on Unknotting Number

Dedicated to the memory of Shouro Kasahara

By Yasutaka NAKANISHI

Let k be an oriented tame knot in a 3-sphere S^3 , and let K be a diagram i.e. the image under a regular projection of S^3 into S^2 . On all diagrams representing k , the minimum number of exchanges of overcrossings and undercrossings required to deform k into a trivial knot is called the *unknotting number* of k , denoted by $u(k)$, and the minimum number of crossings is called the *crossing number* of k , denoted by $cr(k)$ [17].

Observation on diagrams tells us the following fact, where $[x]$ means the maximum integer which is not greater than x .

$$(0.1) \quad 0 \leq u(k) \leq \left[\frac{1}{2} |cr(k) - 1| \right].$$

This inequality is rough but best possible.

We will discuss the relationship between unknotting numbers and numeral algebraic invariants of knots. From Examples 3.1, 2 and 3, we will know the following inequalities are rough but best possible.

$$(1.2) \quad 0 \leq \frac{1}{2} |\sigma(k)| \leq g^*(k) \leq u(k).$$

$$(1.3) \quad 0 \leq m(k) \leq sd(k) \leq u(k).$$

In Section 3.5, we will give counter-examples to [8, Problem 18] i.e. *Is $g^*(k) = g(k')$ for some k' concordant to k ?*

1. Definitions and Inequalities

Throughout this paper, we work in the p.l. category.

Let k be an oriented knot in S^3 . The minimum genus of Seifert surfaces of k is called the 3-genus of k , denoted by $g(k)$ ([7], [20]). R. H. Fox defines the 4-genus of k , denoted by $g^*(k)$ ($h^*(k)$ in [5]).

Proposition 1. ([5]) For any knot k , we have

$$(1.1) \quad 0 \leq g^*(k) \leq g(k).$$

A knot k is called *slice* if $g^*(k) = 0$.

The *signature* of k , denoted by $\sigma(k)$, is known to be an invariant of the knot type [24]. K. Murasugi shows the following in [14].

Theorem 2. ([14]) For any knot k , we have

$$(1.2) \quad 0 \leq \frac{1}{2}|\sigma(k)| \leq g^*(k) \leq u(k).$$

A *surgical description* of k is as follows [18].

Let k be a knot in S^3 , there are disjoint solid tori $T_1, \dots, T_n$ in $S^3 - k$ and a homeomorphism $h : S^3 - ({}^\circ T_1 \cup \dots \cup {}^\circ T_n) \rightarrow S^3 - ({}^\circ T_1 \cup \dots \cup {}^\circ T_n)$ such that

- (i) $h(k)$ is unknotted in S^3 ,
- (ii) the T_i 's are unknotted and pairwise unlinked,
- (iii) $\mathcal{L}k(T_i, k) = \mathcal{L}k(T_i, h(k)) = 0$ for all i ,
- (iv) $h(\partial T_i) = \partial T_i$ and $\mathcal{L}k(\mu_i', T_i) = 1$, where μ_i' is a curve on T_i meridional to T_i and $\mu_i' = h(\mu_i)$.

The minimum number of these solid tori is called the *surgical description number* of k , denoted by $sd(k)$.

Let X be the exterior of k , and let $\tilde{X}_\alpha$ be the universal abelian covering space of X . $H_1(\tilde{X}_\alpha)$ as a $ZH_1(X)$ -module is called the *Alexander invariant* of k , where we take the integral group-ring $ZH_1(X)$ for the Laurent polynomial ring $Z[t, t^{-1}]$ and t is always taken to be represented by the meridian of k . A presentation matrix of $H_1(\tilde{X}_\alpha)$ as a $Z[t, t^{-1}]$ -module is called an *Alexander matrix* of k , denoted by $M_k(t)$. A *minimal Alexander matrix* $M_k^*(t)$ of k is defined to have the minimum size among all square Alexander matrices of k . The number of its size is called the *minor index* of k , denoted by $m(k)$, provided that $m(k) = 0$ if $M_k^*(t)$ is equivalent to (1) as presentation matrices. A surgical view of $H_1(\tilde{X}_\alpha)$ shows the existence of an Alexander matrix of k with size $sd(k)$. We can see that an exchange of an overcrossing and an undercrossing is realized by surgery around a crossing. Hence we have

Theorem 3. *For any knot k , we have*

$$(1.3) \quad 0 \leq m(k) \leq sd(k) \leq u(k).$$

The determinant of an Alexander matrix of k is called the *Alexander polynomial* of k , denoted by $\Delta_k(t)$, which is unique up to units [1].

Theorem 4. ([20]) *For any knot k , we have*

$$(1.4) \quad 0 \leq (\text{the degree of } \Delta_k(t)) \leq 2g(k).$$

2. Deformations preserving $\sigma(k)$ and $m(k)$

Lemma 1. *Let $L = k_1 \cup \cdots \cup k_n$ be an n -components link and let F_i be a Seifert surface of k_i . Let k_0 be a trivial knot such that k_0 and F_i do not meet for every i and that k_0 is contractible in the exterior*

of $F_1 \cup \dots \cup F_n$. We denote by L^* the link which is obtained from L by surgery along k_0 . Then Seifert forms and Alexander invariants of L and L^* are equivalent. In particular, for a knot, this deformation preserves $\sigma(k)$ and $m(k)$.

The proof is obvious, hence we omit it.

Lemma 1'. In Fig. 1, the parallel strings are subarcs of a knot and the shaded area between them is a part of a Seifert surface of this knot respectively. Deformation (a) to (b) preserves $\sigma(k)$ and $m(k)$.

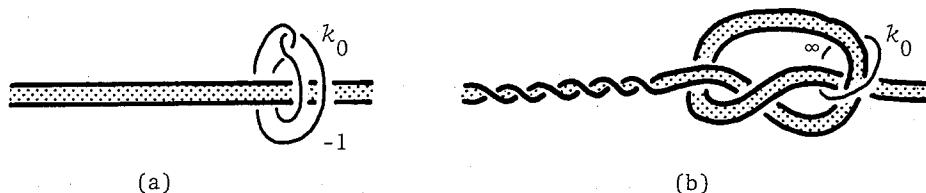


Fig. 1

Proof. In Fig. 1 (b), a surgery curve k_0 is considered as in (a), which satisfies the condition of Lemma 1. This completes the proof.

Lemma 2. ([3], [11], [12]) The join of two prime tangles is a prime knot or a prime 2-components link.

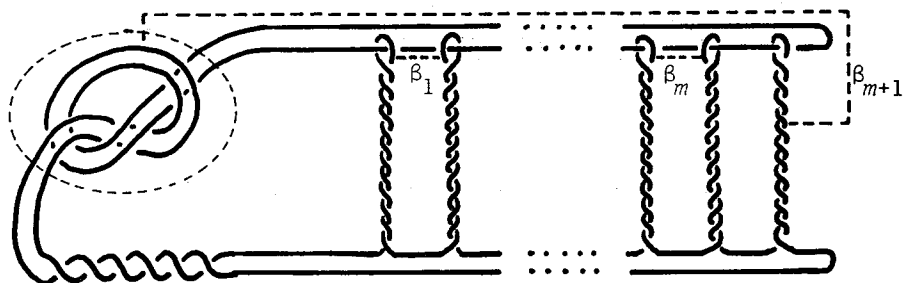
3. Some Examples

3. 1. For any positive integer n , there is a prime knot k such that $g^*(k) = 0$ and $u(k) = n$.

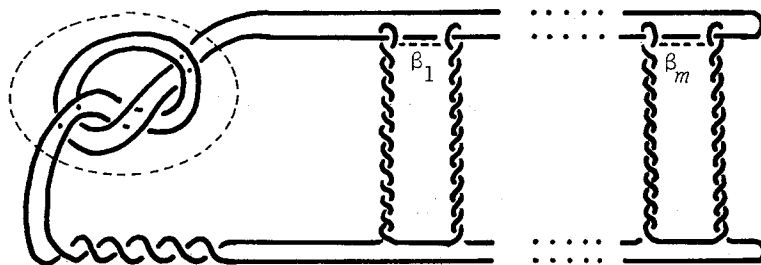
By Lemma 1', the knot k as in Fig. 2 has the same minor index as a connected sum of n copies of the -6 -twisted knot i.e. $m(k) = n$. With

observation on the diagram, we have $u(k) = n$ from (1. 3). On the other hand, ribbon moves at $\beta_1, \dots, \beta_{\lfloor \frac{n+1}{2} \rfloor}$ show that k is slice.

On a similar way we can see the fact: for any pair of positive integers $m \leq n$, there is a prime knot k such that $g^*(k) = m$ and $u(k) = n$.



when $n = 2m+1$ for some integer m



when $n = 2m$ for some integer m

Fig. 2

3. 2. For any positive integer n , there is a prime knot k such that $g^*(k) = u(k) = sd(k) = n$.

By Lemma 1', the knot k as in Fig. 3 has the same minor index and the signature as a connected sum of n copies of the right-hand trefoil knot i.e. $m(k) = \frac{1}{2}|\sigma(k)| = n$. With observation on the diagram, we have $u(k) = g^*(k) = sd(k) = n$ from (1. 2 and 3).

In Sections 3. 1 and 2, the interior and the exterior of a dotted line in Figs. 2 and 3 are prime tangles respectively. From Lemma 2, the above knots are prime.

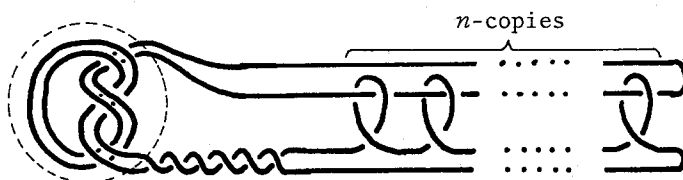


Fig. 3

3. 3. For any positive integer n , there is a prime knot k such that $m(k) = 1$ and $u(k) = n$.

The $(2, 2n+1)$ -torus knot k has $\frac{1}{2}|\sigma(k)| = u(k) = n$ [15]. A group of the torus knot has a one relator group-presentation. Fox's free differential calculus [4] shows $m(k) = 1$.

3. 4. For any positive integer n , there is a prime knot k such that $u(k) = 1$ and $g(k) = n$.

The knot k as in Fig. 4 has the Alexander polynomial with degree $2n$. With observation on the diagram, we have $g(k) = n$ from (1. 4).

We remark this knot is fibered.

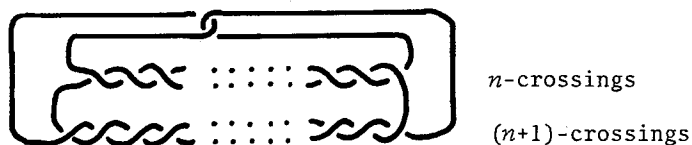


Fig. 4

In Sections 3. 3 and 4, the above knots are 2-bridge, so are prime.

From above arguments, we know (1. 2 and 3) are rough but best possible.

3. 5. For any positive integer n , there is a prime knot k such that $n = g^*(k) < g(k')$ for any k' concordant to k .

H. Kondo [10] and T. Sakai [19] show "For any knot polynomial $\Delta(t)$, there exists a knot k with $u(k) = 1$ and $\Delta_k(t) = \Delta(t)$."

Hence, for any positive integer i , there is a knot k_i with $u(k_i) = 1$ and $\Delta_{k_i}(t) = \frac{t^{2i+1} + 1}{t + 1}$. Since all roots of $\Delta_{k_i}(t)$ are mutually distinct roots of unity, we see $\sigma(k_i) \equiv 2i \pmod{4}$ from J. W. Milnor's arguments in [13]. Hence $|\sigma(k_i)| = 2$ if i is odd, from $u(k_i) = 1$ and (1. 2). Let m be an odd integer greater than $n+1$. There is a connected sum of k_m and $(n-1)$ copies of k_1 (or its mirror image $-k_1$), denoted by k^* , with $|\sigma(k^*)| = 2n$, from the additivity of the signature. Since we easily see $u(k^*) \leq n$, we have $g^*(k^*) = n$ from (1. 2). On the other hand, we know $\Delta_{k^*}(t) = \Delta_{k_m}(t) \cdot \{\Delta_{k_1}(t)\}^{n-1} = (t^{2m} - \dots - t + 1)(t^2 - t + 1)^{n-1}$. Since Alexander polynomials of concordant knots are unique up to $f(t)f(t^{-1})$ -factors [6] where $f(t)$ is a Laurent polynomial with $f(1) = \pm 1$, and $\Delta_{k_m}(t)$ has no $f(t)f(t^{-1})$ -factor, the degree of $\Delta_{k^*}(t)$ for any k^* concordant to k^* is not smaller than $2(m-1)$. From (1. 4), the genus of k^* is not smaller than $m-1$, so is greater than n . Furthermore we can regard k^* as a prime knot from the fact: Any link is concordant to a prime link with the same Alexander invariant (also polynomial) [16]. (Cf. [3].)

Thus we have a required knot.

Remark. If $m(k) = 0$, then $\sigma(k) = 0$. So we cannot decide the unknotting number of such a knot by (1. 2 and 3).

Appendix. The following is a table of Alexander-Briggs' notations [2], half the absolute values of the signatures [22], the 4-genera, the unknotting numbers and the minor indices [15] of prime knots of 9 crossings or less, where 2? means "1 or 2" and 3? means "2 or 3".

3 ₁	1	1	1	1	8 ₈	0	0	2?	1	9 ₈	1	1	2?	1	9 ₂₉	1	1	1	1
4 ₁	0	1	1	1	8 ₉	0	0	1	1	9 ₉	3	3	3	1	9 ₃₀	0	1	1	1
5 ₁	2	2	2	1	8 ₁₀	1	1	2?	1	9 ₁₀	2	2	3?	1	9 ₃₁	1	2?	2?	1
5 ₂	1	1	1	1	8 ₁₁	1	1	1	1	9 ₁₁	2	2	2	1	9 ₃₂	1	2?	2?	1
6 ₁	0	0	1	1	8 ₁₂	0	1	2?	1	9 ₁₂	1	1	1	1	9 ₃₃	0	1	1	1
6 ₂	1	1	1	1	8 ₁₃	0	1	1	1	9 ₁₃	2	2	3?	1	9 ₃₄	0	1	1	1
6 ₃	0	1	1	1	8 ₁₄	1	1	1	1	9 ₁₄	0	1	1	1	9 ₃₅	1	1	3?	2
7 ₁	3	3	3	1	8 ₁₅	2	2	2	1	9 ₁₅	1	1	2?	1	9 ₃₆	2	2	2	1
7 ₂	1	1	1	1	8 ₁₆	1	2?	2?	1	9 ₁₆	3	3	3	1	9 ₃₇	0	1	2	2
7 ₃	2	2	2	1	8 ₁₇	0	1	1	1	9 ₁₇	1	2?	2?	1	9 ₃₈	2	2	3?	2?
7 ₄	1	1	2?	1	8 ₁₈	0	2?	2	2	9 ₁₈	2	2	2	1	9 ₃₉	1	1	1	1
7 ₅	2	2	2	1	8 ₁₉	3	3	3	1	9 ₁₉	0	1	1	1	9 ₄₀	1	2?	2	2
7 ₆	1	1	1	1	8 ₂₀	0	0	1	1	9 ₂₀	2	2	2	1	9 ₄₁	0	0	2	2
7 ₇	0	1	1	1	8 ₂₁	1	1	1	1	9 ₂₁	1	1	1	1	9 ₄₂	1	1	1	1
8 ₁	0	1	1	1	9 ₁	4	4	4	1	9 ₂₂	1	1	1	1	9 ₄₃	2	2	2	1
8 ₂	2	2	2	1	9 ₂	1	1	1	1	9 ₂₃	2	2	2	1	9 ₄₄	0	1	1	1
8 ₃	0	1	2?	1	9 ₃	3	3	3	1	9 ₂₄	0	1	1	1	9 ₄₅	1	1	1	1
8 ₄	1	1	2?	1	9 ₄	2	2	2	1	9 ₂₅	1	1	2?	1	9 ₄₆	0	0	2	2
8 ₅	2	2	2	1	9 ₅	1	1	2?	1	9 ₂₆	1	1	1	1	9 ₄₇	1	2?	2	2
8 ₆	1	1	2?	1	9 ₆	3	3	3	1	9 ₂₇	0	0	1	1	9 ₄₈	1	2?	2	2
8 ₇	1	1	1	1	9 ₇	2	2	2	1	9 ₂₈	1	1	1	1	9 ₄₉	2	2	3?	2

References

- [1] J. W. Alexander: Topological invariants of knots and links, Trans. Amer. Math. Soc., 30 (1928), 275-306.
- [2] J. W. Alexander and G. B. Briggs: On types of knotted curves, Ann. of Math., 28 (1927), 562-586.
- [3] S. A. Bleiler: Realizing concordant polynomials with prime knots, preprint.
- [4] R. H. Crowell and R. H. Fox: An Introduction to Knot Theory, Ginn and Co., Boston, Mass., 1963.
- [5] R. H. Fox: Some problems in knot theory, Topology of 3-Manifolds and Related Topics (Proc., The Univ. of Georgia Inst., 1961), Prentice-Hall, Englewoods Cliffs, New Jersey, 1962, 120-167.
- [6] R. H. Fox and J. Milnor: Singuralities of 2-spheres in 4-space and equivalence of knots, unpublished. Cf. Bull. Amer. Math. Soc., 63 (1957), 406.
- [7] F. Frankl and L. Pontrjagin: Ein Knotensatz mit Anwendung auf die Dimensionstheorie, Math. Ann., 102 (1930), 785-789.
- [8] McA. Gordon (ed.): Problems, Knot Theory (Proc., Plans-sur-Bex, 1977), Lecture Notes in Math., 685, Springer-Verlag, 1978, 309-311.
- [9] S. Kinoshita: On Wendt's theorem of knots, Osaka Math. J., 9 (1957), 61-66.
- [10] H. Kondo: Knots with the unknotting number 1 and their Alexander polynomials, Osaka J. Math., 16 (1979), 551-559.
- [11] R. Kirby and W. B. R. Lickorish: Prime knots and concordance, Math. Proc. Camb. Phil. Soc., 86 (1979), 437-441.
- [12] W. B. R. Lickorish: Prime tangles and knots, preprint.
- [13] J. W. Milnor: Infinite cyclic coverings, Conference on the Topology of Manifolds, Prindle, Weber and Schmidt, Boston, Mass., 1968, 115-133.
- [14] K. Murasugi: On a certain numerical invariant of link types, Trans. Amer. Math. Soc., 117 (1965), 387-422.

- [15] Y. Nakanishi: Alexander invariants of links (in Japanese), Master Thesis, Kobe Univ., 1980.
- [16] Y. Nakanishi: Prime links, concordance and Alexander invariants, Math. Sem. Notes, Kobe Univ., 8 (1980), 561-568.
- [17] K. Reidemeister: Knotentheorie, Ergebnisse der Mathematik, 1, Springer-Verlag, Berlin, 1932.
- [18] D. Rolfsen: A surgical view of Alexander's polynomial, Geometric Topology (Proc., Park City, 1974), Lecture Notes in Math., 438, Springer-Verlag, 1974, 415-423.
- [19] T. Sakai: A remark on the Alexander polynomials of knots, Math. Sem. Notes, Kobe Univ., 5 (1977), 451-456.
- [20] H. Seifert: Über das Geschlecht von Knoten, Math. Ann., 110 (1934), 571-592.
- [21] T. Shibuya: Some relations among various numerical invariants for links, Osaka J. Math., 11 (1974), 313-322.
- [22] Y. Shinohara: On the signature of knots and links, Ph.D. Dissertation, Florida State Univ., Tallahassee, Fla., 1969.
- [23] G. Torres: On the Alexander polynomial, Ann. of Math., 57 (1953), 57-89.
- [24] H. F. Trotter: Homology of group systems with applications to knot theory, Ann. of Math., 76 (1962), 464-498.
- [25] H. Wendt: Die gordische Auflösung von Knoten, Math. Z., 42 (1937), 680-696.

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