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Centrality in Lattice Ordered Autometrized Algebras

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J. D. H. Smith [1] introduced the notion of centrality in an algebra A as follows:

Definition. Let A be an algebra, let β, α be congruences on A , and let $(\alpha \mid \beta)$ be a congruence on β . Then α is said to *centralise* β by means of the *centring congruence* $(\alpha \mid \beta)$ iff the following conditions are satisfied:

(C0) : $(x, y)(\alpha \mid \beta)(x', y')$ implies $x \alpha x'$.

(C1) : For every $(x, y) \in \beta$, $\pi^0 : (x, y)^{(\alpha \mid \beta)} \rightarrow x$; $(x', y') \mapsto x'$ bijects.

(C2) : The following three conditions are satisfied:

(RR) : For every $(x, y) \in \alpha$, $(x, x)(\alpha \mid \beta)(y, y)$.

(RS) : $(x, y)(\alpha \mid \beta)(x', y')$ implies $(y, x)(\alpha \mid \beta)(y', x')$.

(RT) : $(x, y)(\alpha \mid \beta)(x', y')$ and $(y, z)(\alpha \mid \beta)(y', z')$ imply $(x, z)(\alpha \mid \beta)(x', z')$.

Conditions (RR), (RS), and (RT) respectively are known as *respect for the reflexivity, symmetry, and transitivity* of β . (C2) is called *respect for equivalence*. Smith [1] gave two examples as follows to show how the conditions of the above definition work. Firstly, let A be a group, and let β and α be congruences on A and let B and C be

the corresponding normal subgroups of A respectively. Then α centralises β by means of a centreing congruence $(\alpha \mid \beta)$ if and only if C centralises B in the usual grouptheoretic sense, i.e., iff $bc = cb$ for all c in C and for all b in B . Secondly, let A be a ring and let α, β be congruences on A and C, B be the corresponding ideals of A respectively. Then, α centralises β by means of a centreing congruence $(\alpha \mid \beta)$ if and only if $BC + CB = 0$. He also proved that centreing congruences are unique for any algebra belonging to a Mal'cev variety.

In [2], we have introduced the notion of a representable autometrized algebra $A = (A, +, \leq, *)$ and observed that there is a 1-1 correspondence between the ideals of A and the congruences of A . In this paper, $A = (A, +, \leq, *)$ stands for a representable autometrized algebra satisfying the identity (R) : $a * (a \cap b) + a \cap b = a$ for all a, b in A . Here, by an ideal I of A , we mean a nonempty subset I of A satisfying (I1) $a, b \in I$ implies $a + b \in I$ and (I2) $b \in I, a \in A$ and $a * 0 \leq b * 0$ implies $a \in I$; and by a congruence θ on A (see Lemma 2, [4]), we mean an equivalence relation θ on A satisfying the substitution property for all the operations $+, \cup, \cap$ and $*$ on A . In [3], we have proved that the ternary operation f on A defined by $f(x, y, z) = [x * (x \cap y) + x \cap z] \cup [z * (z \cap y) + z \cap x]$ for all x, y, z in A , is a Mal'cev operation on A , i.e., f satisfies the conditions $f(x, x, z) \equiv z$ and $f(x, z, z) \equiv x$, for all x, z in A .

In the present paper, we prove the following

Theorem. Let β, α be congruences on A , and let B and C be the

ideals of A corresponding to β and α respectively. Then, α centralizes β by means of a centreing congruence $(\alpha \mid \beta)$ if and only if $B \cap C = \{0\}$.

Remark. As the class of representable autometrized algebras satisfying the identity (R) is a wide class containing properly the class of commutative DR ℓ -semigroups which includes commutative ℓ -groups, Brouwerian algebras and Boolean algebras as special cases, the above theorem is applicable to commutative DR ℓ -semigroups and hence to commutative ℓ -groups, Brouwerian algebras and Boolean algebras. It can be proved exactly on the same lines as in [1], that if α and β are congruences on a representable autometrized algebra satisfying (R), such that α centralizes β by means of a centreing congruence $(\alpha \mid \beta)$, then that centreing congruence $(\alpha \mid \beta)$ is unique.

Proof of the theorem. $a \leq a * 0$ for all a in A , since $a = a * (a \cap 0) + a \cap 0 \leq a * 0 + 0 = a * 0$ (since $a \cap x$ is a contraction).

First, assume that α centralises β by means of a centreing congruence $(\alpha \mid \beta)$. Let $b \in B \cap C$. Then $(b, 0) \in \beta$, $(b, 0) \in \alpha$ and $(b, b) \in \beta$.

Now, we have

(1) : $(b, 0)(\alpha \mid \beta)(b, 0)$ by the reflexivity of $(\alpha \mid \beta)$

(2) : $(b, b)(\alpha \mid \beta)(0, 0)$ by (RR)

(3) : $(b, b)(\alpha \mid \beta)(b, b)$ by the reflexivity of $(\alpha \mid \beta)$.

Since $(\alpha \mid \beta)$ has the substitution property for all the operations $+$, $\cup$, $\cap$ and $*$, we have

(4) : $(b, b \cap 0)(\alpha \mid \beta)(b \cap 0, 0)$ from (1) and (2)

(5) : $(0, b * 0)(\alpha \mid \beta)(b * 0, 0)$ from (1) and (2)

(6) : $(b \cap 0, b)(\alpha \mid \beta)(b, b \cap 0)$ from (3) and (5) ($b \leq b * 0$)
 From (4) and (6), it follows that $(b \cap 0, 0) \in (b, b \cap 0)^{(\alpha \mid \beta)}$ and
 $(b \cap 0, b) \in (b, b \cap 0)^{(\alpha \mid \beta)}$. Applying (C1), we have $b = 0$. Hence
 $B \cap C = \{0\}$.

Conversely, assume that $b \cap c = \{0\}$. Define a relation $(\alpha \mid \beta)$ on
 β as follows:

$(x, y)(\alpha \mid \beta)(x', y')$ if and only if $x * x' = y * y'$ and $x * x' \in C$. Then, $(x, y)(\alpha \mid \beta)(x, y)$ for all $(x, y) \in \beta$, since $x * x = 0 = y * y$ and $0 \in C$. So, $(\alpha \mid \beta)$ is reflexive. If $(x, y)(\alpha \mid \beta)(x', y')$, then, $x * x' = y * y'$ and $x * x' \in C$, so that $x' * x = y' * y$ and $x' * x \in C$ and hence follows that $(x', y')(\alpha \mid \beta)(x, y)$. Thus, $(\alpha \mid \beta)$ is symmetric. To prove the transitivity of $(\alpha \mid \beta)$, assume that $(x, y)(\alpha \mid \beta)(x', y')$ and $(x', y')(\alpha \mid \beta)(x'', y'')$. Then, $x * x' = y * y'$, $x * x' \in C$ and $x' * x'' = y' * y''$, $x' * x'' \in C$. But, by semiregularity of $*$, we have $(x * x'') * 0 = x * x'' \leq x * x' + x' * x'' = (x * x' + x' * x'') * 0$, where $(x * x' + x' * x'') \in C$ (since C is an ideal of A). Hence $x * x'' \in C$. In a similar manner, we have $y * y'' \in C$. But,
 $[(x * x'') * (y * y'')] * 0 = (x * x'') * (y * y'') \leq (x * x'') * 0 + 0 * (y * y'') = (x * x'') + (y * y'') = [(x * x'') + (y * y'')] * 0$, where $(x * x'' + y * y'') \in C$ (since C is an ideal). So, we have $[(x * x'') * (y * y'')] \in C$. Since $(x, y) \in \beta$ and $(x'', y'') \in \beta$, we have $x * y \in B$ and $x'' * y'' \in B$ so that $(x * y + x'' * y'') \in B$ (since B is an ideal of A). We, also, have $[(x * x'') * (y * y'')] * 0 = (x * x'') * (y * y'') \leq x * y + x'' * y''$ (since $a \mapsto a * x$ is a contraction) $= [(x * y) + (x'' * y'')] * 0$. Since B is an ideal, this implies $[(x * x'') * (y * y'')] \in B$. Thus,
 $[(x * x'') * (y * y'')] \in B \cap C = 0$. Hence $x * x'' = y * y''$ and $x * x''$

$\in C$. Hence, we have $(x, y)(\alpha \mid \beta)(x'', y'')$. Thus, $(\alpha \mid \beta)$ is transitive also and hence $(\alpha \mid \beta)$ is an equivalence relation on β . To show that $(\alpha \mid \beta)$ is a congruence on β , let θ denote any one of the operations $+$, $\cup$, $\cap$, and $*$ on β . Let $(x_1, y_1)(\alpha \mid \beta)(x_2, y_2)$ and $(x_3, y_3)(\alpha \mid \beta)(x_4, y_4)$. Then, we have $x_1 * x_2 = y_1 * y_2$, $x_3 * x_4 = y_3 * y_4$, $x_1 * x_2 \in C$ and $x_3 * x_4 \in C$. We know that $a \mapsto a \theta x$ is a contraction for each $a \in A$. Using this, we have $[(x_1 \theta x_3) * (x_2 \theta x_4)] * 0 = (x_1 \theta x_3) * (x_2 \theta x_4) \leq x_1 * x_2 + x_3 * x_4 = [x_1 * x_2 + x_3 * x_4] * 0$, where $(x_1 * x_2 + x_3 * x_4) \in C$. Since C is an ideal, it follows that $[(x_1 \theta x_3) * (x_2 \theta x_4)] \in C$. Similarly, we have $[(y_1 \theta y_3) * (y_2 \theta y_4)] \in C$. Hence, we have $\{[(x_1 \theta x_3) * (x_2 \theta x_4)] + [(y_1 \theta y_3) * (y_2 \theta y_4)]\} \in C$. Using the triangle inequality and semiregularity of $*$, since C is an ideal, we obtain $\{[(x_1 \theta x_3) * (x_2 \theta x_4)] * [(y_1 \theta y_3) * (y_2 \theta y_4)]\} \in C$. Also, using the contraction property of the mappings $a \mapsto a \theta x$ and $a \mapsto a * x$, we have $\{[(x_1 \theta x_3) * (x_2 \theta x_4)] * [(y_1 \theta y_3) * (y_2 \theta y_4)]\} * 0 = \{(x_1 \theta x_3) * (x_2 \theta x_4)\} * \{(y_1 \theta y_3) * (y_2 \theta y_4)\} \leq (x_1 \theta x_3) * (y_1 \theta y_3) + (x_2 \theta x_4) * (y_2 \theta y_4) \leq (x_1 * y_1 + x_3 * y_3) + (x_2 * y_2 + x_4 * y_4) = (x_1 * y_1 + x_2 * y_2 + x_3 * y_3 + x_4 * y_4) * 0$, where $(x_1 * y_1 + x_2 * y_2 + x_3 * y_3 + x_4 * y_4) \in B$. Since B is an ideal, we have $\{[(x_1 \theta x_3) * (x_2 \theta x_4)] * [(y_1 \theta y_3) * (y_2 \theta y_4)]\} \in B$. Hence $\{[(x_1 \theta x_3) * (x_2 \theta x_4)] * [(y_1 \theta y_3) * (y_2 \theta y_4)]\} \in B \cap C = 0$. This implies $(x_1 \theta x_3) * (x_2 \theta x_4) = (y_1 \theta y_3) * (y_2 \theta y_4)$, where, $(x_1 \theta x_3) * (x_2 \theta x_4) \in C$. Hence, follows that $(x_1 \theta x_3, y_1 \theta y_3)(\alpha \mid \beta)(x_2 \theta x_4, y_2 \theta y_4)$, i.e., $((x_1, y_1) \theta (x_3, y_3))(\alpha \mid \beta)((x_2, y_2) \theta (x_4, y_4))$. This implies $(\alpha \mid \beta)$ has substitution property for all the operations $+$, $\cup$, $\cap$ and $*$. Hence $(\alpha \mid \beta)$ is a congruence

on B .

To show that α centralises β by $(\alpha \mid \beta)$, let us verify the condition (C0), (C1) and (C2) in the definition.

(C0) : $(x, y)(\alpha \mid \beta)(x', y')$ implies $x * x' = y * y'$ and $x * x' \in C$, which implies $(x, x') \in \alpha$ or $x \alpha x'$.

(C1) : Let $(x, y) \in \beta$ and let $\pi^0 : (x, y)(\alpha \mid \beta) \rightarrow x^\alpha$ be defined by $(x', y') \mapsto x'$.

π^0 is surjective: Let $x' \in x^\alpha$. Then $(x, x') \in \alpha$ and so $x * x' \in C$. Put $y' = f(y, x * x')$, where f is the Mal'cev operation on A mentioned in the introduction. We have, $(y, y) \in \alpha$, $(x', x) \in \alpha$ and $(x', x) \in \alpha$. Hence, $(f(y, x, x'), f(y, x, x)) \in \alpha$, i.e., $(y', y) \in \alpha$. Hence $y * y' \in C$. So, we have $[(x * x') * (y * y')] \in C$ ($\because C$ is an ideal). Since, $(y, x) \in \beta$, $(x, x) \in \beta$ and $(x', x') \in \beta$, we have $(f(y, x, x'), f(x, x, x')) \in \beta$, i.e., $(y', x') \in \beta$. Hence $x' * y' \in B$. Since $(x, y) \in \beta$, we have $x * y \in B$. Since B is an ideal, and $[(x * x') * (y * y')] 0 = (x * x') * (y * y') \leq x * y + x' * y' = (x * y + x' * y') * 0$, where $(x * y + x' * y') \in B$, we have $(x * x') * (y * y') \in B$. Hence, $[(x * x') * (y * y')] \in B \cap C = \{0\}$. This implies $x * x' = y * y'$, where $x * x' \in C$. Hence, we have $(x, y)(\alpha \mid \beta)(x', y')$ and $\pi^0(x', y') = x'$. Thus, π^0 is surjective.

π^0 is injective: Assume that $(x', y') \in (x, y)(\alpha \mid \beta)$ and $(x'', y'') \in (x, y)(\alpha \mid \beta)$ and $\pi^0(x', y') = \pi^0(x'', y'')$. Then, we have $x' = x''$, $(x, y)(\alpha \mid \beta)(x', y')$ and $(x, y)(\alpha \mid \beta)(x'', y'')$. Since $(\alpha \mid \beta)$ is a congruence, it follows that $(x', y')(\alpha \mid \beta)(x'', y'')$. This implies $x' * x'' = y' * y''$ and $x' * x' \in C$, i.e., $y' * y'' = 0$, which implies $y' = y''$. Hence, π^0 is injective. Thus π^0 is bijective.

(RR) : $(x, y) \in \alpha$ implies $x * y \in C$ which implies $(x, x)(\alpha \mid \beta)$
 (y, y) (since $x * y = x * y$ and $x * y \in C$).

(RS) : $(x, y)(\alpha \mid \beta)(x', y')$ implies $x * x' = y * y'$ and $x * x' \in C$,
 which implies $y * y' = x * x'$ and $y * y' \in C$, which implies (y, x)
 $(\alpha \mid \beta)(y', x')$.

(RT) : Assume that $(x, y)(\alpha \mid \beta)(x', y')$ and $(y, z)(\alpha \mid \beta)(y', z')$.
 Then, $x * x' = y * y' = z * z'$ and $x * x' \in C$. Hence, we have $x * x' =$
 $z * z'$ and $x * x' \in C$, so that $(x, z)(\alpha \mid \beta)(x', z')$.
 Thus, $(\alpha \mid \beta)$ satisfies the conditions (C0), (C1) and (C2) in the defin-
 ition. Hence α centralises β by means of the centring congruence
 $(\alpha \mid \beta)$. This completes the proof.

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