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UNIMODALITY CONDITIONS OF THE DISTRIBUTION OF A MIXTURE
OF TWO DISTRIBUTIONS

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Conditions for which the distribution of a mixture of two normal distributions is unimodal were investigated by Eisenberger [1]. To the problem of the pooling of two samples, Takeshita, Kasagi and Yanagawa [4], and Inada and Kakiuchi [2] employed unimodality conditions of the distribution of the pooled samples. We may be especially interested in whether a distribution is unimodal or bimodal because the information contained in this fact alone may be more important than the parameters of the actual distribution.

In this paper we consider the distribution of a mixture of two distributions without the assumption of the normality and investigate a critical value of the difference between means of two distributions for which the distribution is unimodal for any proportions p and $1-p$, $0 < p < 1$. In section 1, when only means are different, the critical value which is independent of p is obtained. In section 2, when both means and variances are different, some results in section 1 are extended, and a condition of the shape of the probability density function of the individual distribution for which the distribution is never unimodal is obtained. For the normal,

logistic, double exponential and t distributions, the critical values are compared in section 3.

1. Only means are different

Let $F(x)$ be a cumulative distribution function (cdf) which is symmetric and unimodal at $x=0$, and $f(x)$ be its probability density function (pdf).

We shall consider unimodality conditions for the cdf $G(x;p)$ of a mixture which is given by

$$G(x;p) = (1-p)F((x-\mu_1)/\sigma) + pF((x-\mu_2)/\sigma) \quad (1)$$

where μ_1 and μ_2 , $-\infty < \mu_1, \mu_2 < \infty$, are location parameters, $\sigma > 0$ is a scale parameter and $0 < p < 1$.

Now instead of (1), it is sufficient to discuss the unimodality of the cdf

$$G(x;p) = (1-p)F(x) + pF(x-\mu_\sigma) \quad (2)$$

where

$$\mu_\sigma = (\mu_2 - \mu_1)/\sigma. \quad (3)$$

In this paper we assume that $f(x)$ is continuous on the real line and has the continuous derivative $f'(x)$ on the real line except for $x=0$.

If $f'(0)$ exists, then it is called by Type A and if not, it is called by Type B.

It is noted that for the pdf of Type A $f'(0)=0$ and for the pdf of Type B $f'(x) < 0$ as $x \rightarrow +0$.

In this section we shall consider the following condition for the shape of the pdf $f(x)$.

Condition 1. There exists the value of x' , $0 < x' < \infty$, such that $f'(x)$ is nonincreasing on the interval $(0, x']$ and nondecreasing on the interval (x', ∞) .

For example, the pdf's of the normal, logistic and t distributions of Type A satisfy Condition 1 but the pdf of the double exponential distribution of Type B does not satisfy Condition 1.

From Condition 1, we obtain that there exists values of x_1 and x_2 , $0 \leq x_1 < x_2 \leq \infty$, such that for the pdf of Type A

$$\begin{aligned} f'(x) &= 0 & \text{if } |x| \leq x_1 \text{ or } |x| \geq x_2 \\ &\neq 0 & \text{if otherwise} \end{aligned} \quad (4)$$

or for the pdf of Type B

$$\begin{aligned} f'(x) &= 0 & \text{if } |x| \geq x_2 \\ &\neq 0 & \text{if otherwise.} \end{aligned} \quad (5)$$

Now put

$$h(x) = -f'(x - \mu_0)/f'(x). \quad (6)$$

Lemma A. Suppose that Condition 1 is satisfied for the pdf $f(x)$ of Type A. Then, $G(x; p)$ is unimodal for all p , $0 < p < 1$, if and only if the following conditions (i) or (ii) are satisfied:

- (i) $|\mu_0| \leq 2x_1$.
- (ii) $2x_1 < |\mu_0| \leq x_1 + x_2$ and $h(x)$ is nonincreasing on the interval $(x_1, \mu_0 - x_1)$.

Proof. Denote $G'(x; p)$ and $G''(x; p)$ by the first and second derivatives of $G(x; p)$ respectively. If $\mu_0 = 0$, $G(x; p)$ is evidently unimodal for

all p . Suppose $\mu_\sigma > 0$. Then, $G(x;p) \geq 0$ for $x < 0$ and $G(x;p) \leq 0$ for $x > \mu_\sigma$. As $G'(x;p)$ is continuous at $x=0$ and μ_σ , we are now interested only in $G''(x;p)$ for $0 < x < \mu_\sigma$. Thus, $G(x;p)$ is unimodal for all p if and only if for any fixed p there exists the value of x' , $0 \leq x' \leq \mu_\sigma$, such that $G''(x;p) \geq 0$ for $0 < x \leq x'$ and $G''(x;p) \leq 0$ for $x' < x < \mu_\sigma$.

At first suppose $\mu_\sigma \leq 2x_1$. Using the symmetry and unimodality of $f(x)$ at $x=0$, we obtain that $G''(x;p) = 0$ for $0 < x < \mu_\sigma$ if $\mu_\sigma \leq x_1$, and $G''(x;p) \geq 0$ for $0 < x \leq x_1$ and $G''(x;p) \leq 0$ for $x_1 < x < \mu_\sigma$ if $x_1 < \mu_\sigma \leq 2x_1$. Thus, (i) implies that $G(x;p)$ is unimodal for all p .

Secondly suppose $2x_1 < \mu_\sigma \leq x_1 + x_2$. Then, $G''(x;p) \geq 0$ for $0 < x \leq x_1$ and $G''(x;p) \leq 0$ for $\mu_\sigma - x_1 \leq x < \mu_\sigma$. Putting $G''(x;p) = 0$, we obtain $h(x) = p/(1-p)$. This right term takes on all finite positive values exactly once on the interval $0 < p < 1$. For $x_1 < x < \mu_\sigma - x_1$, $h(x) > 0$. If $h(x)$ is nonincreasing on the interval $(x_1, \mu_\sigma - x_1)$, then for any fixed p there exists x' , $x_1 \leq x' \leq \mu_\sigma - x_1$, such that $h(x) \geq p/(1-p)$ for $x_1 < x \leq x'$ and $h(x) \leq p/(1-p)$ for $x' < x < \mu_\sigma - x_1$. Thus, (ii) implies that $G(x;p)$ is unimodal for all p .

Finally we show that (i) or (ii) are necessary conditions. If $2x_1 < \mu_\sigma \leq x_1 + x_2$ and $h(x)$ is not non increasing on the interval $(x_1, \mu_\sigma - x_1)$, there exists p and x' , $x_1 < x' < \mu_\sigma - x_1$, such that $h(x) < p/(1-p)$ on some subinterval of (x_1, x') and $h(x) > p/(1-p)$ on some subinterval of $(x', \mu_\sigma - x_1)$. Then, there exists p such that $G(x;p)$ is not unimodal. Suppose $x_1 + x_2 < \mu_\sigma$. Since $f'(x) < 0$ and $f'(\mu_\sigma - x) = 0$ on the interval $(x_1, \min(x_2, \mu_\sigma - x_2))$, $G''(x;p) < 0$ on this interval. Also, since $f'(x) = 0$ and $f'(\mu_\sigma - x) < 0$ on the interval $(\max(x_2, \mu_\sigma - x_2), \mu_\sigma - x_1)$, $G''(x;p) > 0$ on this interval. This implies that $G(x;p)$ is not unimodal for all p .

It is easy to show that the same results are obtained for $\mu_\sigma < 0$, which completes the proof.

Now let x_0 be the greatest value of x' , satisfying Condition 1.

Put

$$K(x) = f''(x)/f'(x) + f''(2x'_0 - x)/f'(2x'_0 - x) \quad (7)$$

where

$$x'_0 = \min(x_0, (x_1 + x_2)/2). \quad (8)$$

Theorem 1A. Suppose that Condition 1 is satisfied for the pdf $f(x)$ of Type A. Then, the following results are derived:

- (i) If $|\mu_\sigma| \leq x_0$, then $G(x;p)$ is unimodal for all p , $0 < p < 1$.
- (ii) Suppose that there exists the derivative $f''(x)$ of $f'(x)$ on the interval $(x_1, 2x'_0 - x_1)$, and that $f''(x)/f'(x)$ is nonincreasing and $K(x)$ is nonnegative on this interval. If $|\mu_\sigma| \leq 2x'_0$, then $G(x;p)$ is unimodal for all p , $0 < p < 1$.
- (iii) If $|\mu_\sigma| > 2x'_0$, then there exists p , $0 < p < 1$, such that $G(x;p)$ is not unimodal.

Proof. At first suppose $0 \leq \mu_\sigma \leq x_0$. Since $f'(x)$ is negative and nonincreasing on the interval $(x_1, \mu_\sigma - x_1)$, $h(x)$ is nonincreasing on this interval and hence (i) is proved from Lemma A.

Secondly suppose $x_0 < \mu_\sigma \leq 2x'_0$. By the assumption of (ii), we have

$$\begin{aligned} h'(x) &= - (f'(\mu_\sigma - x)/f'(x)) (f''(x)/f'(x) + f''(\mu_\sigma - x)/f'(\mu_\sigma - x)) \\ &\leq - (f'(\mu_\sigma - x)/f'(x)) (f''(x)/f'(x) + f''(2x'_0 - x)/f'(2x'_0 - x)) \\ &\leq 0 \end{aligned}$$

on the interval $(x_1, \mu_\sigma - x_1)$. Thus, (ii) is proved from Lemma A and (i).

Finally suppose $\mu_\sigma > 2x'_0$. If $x_0 \geq (x_1 + x_2)/2$, we have $\mu_\sigma > x_1 + x_2$ and hence (iii) is proved from Lemma A. If $x_0 < (x_1 + x_2)/2$, since $x_0 < ((\mu_\sigma - x_1) + x_1)/2$, we obtain that there exists x' , $x_0 < x'$, such that $f'(x)$ is increasing and $f'(\mu_\sigma - x)$ is nonincreasing on the interval $[x_0, x']$. Thus $h(x)$ is increasing on this interval. Therefore, (iii) is proved from Lemma A.

Suppose that the assumption of (ii) in the above theorem is satisfied. Then, it is noted that $G(x;p)$ is unimodal for all p , $0 < p < 1$, if and only if $|\mu_\sigma| \leq 2x'_0$.

We easily obtain the following results for the pdf $f(x)$ of Type B by the same manner as was discussed about the pdf of Type A.

Lemma B. Suppose that Condition 1 is satisfied for the pdf $f(x)$ of Type B. Then, $G(x;p)$ is unimodal for all p , $0 < p < 1$, if and only if $|\mu_\sigma| < x_2$ and $h(x)$ is nonincreasing on the interval $(0, \mu_\sigma)$.

Proof. Now suppose $\mu_\sigma = x_2$. Then, $h(x) \rightarrow +0$ as $x \rightarrow +0$ and $h(x) \rightarrow +\infty$ as $x \rightarrow \mu_\sigma - 0$. Since $h(x)$ is positive and continuous on the interval $(0, \mu_\sigma)$, $h(x)$ is not nonincreasing on this interval. Thus, there exists p , $0 < p < 1$, such that $G(x;p)$ is not unimodal. For the other parts $0 < \mu_\sigma < x_2$ and $x_2 < \mu_\sigma$, we can easily obtain the same results as those in the proof of Lemma A by letting $x_1 = 0$. Therefore, as was shown in Lemma A, we obtain this lemma.

Theorem 1B. Suppose that Condition 1 is satisfied for the pdf $f(x)$ of Type B. Then, the following results are derived:

- (i) If $|\mu_\sigma| \leq x_0$, then $G(x;p)$ is unimodal for all p , $0 < p < 1$.
- (ii) Suppose that the assumption of (ii) in Theorem 1A is satisfied by letting $x_1 = 0$ and $x'_0 = x''_0 \equiv \min(x_0, x_2/2)$. If $|\mu_\sigma| \leq 2x''_0$ when $2x_0 < x_2$, or if $|\mu_\sigma| < 2x''_0$ when $2x_0 \geq x_2$, then $G(x;p)$ is unimodal for all p , $0 < p < 1$.
- (iii) If $|\mu_\sigma| > 2x''_0$ when $2x_0 < x_2$, or if $|\mu_\sigma| \geq 2x''_0$ when $2x_0 \geq x_2$, then there exists p , $0 < p < 1$, such that $G(x;p)$ is not unimodal.

Proof. From Lemma B, if $|\mu_\sigma| = x_2$, $G(x;p)$ is not unimodal for all p . Eliminating the case of $|\mu_\sigma| = x_2$, the proof is the same as that described for Type A in Theorem 1A by letting $x_1 = 0$. Thus, the theorem is proved.

From Theorem 1A and 1B, the following corollary is the immediate consequence.

Corollary. Suppose that $f'(x) \neq 0$ for $0 < |x| \leq 2x_0$ and Condition 1 is satisfied. Suppose that the assumption of (ii) in Theorem 1A is satisfied by letting $x_1 = 0$ and $x'_0 = x_0$. Then, $G(x;p)$ is unimodal for all p , $0 < p < 1$, if and only if $|\mu_\sigma| \leq 2x_0$.

2. Both means and variances are different

It is possible to extend some results in section 1 to the case that both means and variances of two distributions are different.

We shall now consider the cdf

$$G(x;p) = (1-p)F((x-\mu_1)/\sigma_1) + pF((x-\mu_2)/\sigma_2), \quad 0 < p < 1. \quad (9)$$

Theorem 2. Suppose that $f'(x) \neq 0$ for $0 < |x|$ and Condition 1 is satisfied. Then, the following results are derived:

(i) If $|\mu_1 - \mu_2| \leq x_0 \min(\sigma_1, \sigma_2)$, then $G(x;p)$ is unimodal for all p , $0 < p < 1$.

(ii) If $|\mu_1 - \mu_2| > x_0(\sigma_1 + \sigma_2)$, then there exists p , $0 < p < 1$, such that $G(x;p)$ is not unimodal.

Proof. Now we consider

$$G(x;p) = (1-p)F(x) + pF((\sigma_1/\sigma_2)x - (\mu_2 - \mu_1)/\sigma_2)$$

instead of (9).

At first we prove (i). Suppose $0 < \mu_2 - \mu_1 \leq x_0 \min(\sigma_1, \sigma_2)$. As was shown in Lemma A, it is sufficient to show that $h(x) = f'((\mu_2 - \mu_1)/\sigma_2 - (\sigma_1/\sigma_2)x) / f'(x)$ is nonincreasing on the interval $(0, (\mu_2 - \mu_1)/\sigma_1)$. Since $(\mu_2 - \mu_1)/\sigma_1 < x_0$ and $(\mu_2 - \mu_1)/\sigma_2 < x_0$, this is immediately obtained.

Next we prove (ii). Suppose $\mu_2 - \mu_1 > x_0(\sigma_1 + \sigma_2)$. Since $x_0 < (\mu_2 - \mu_1)/\sigma_2 - (\sigma_1/\sigma_2)x_0$, there exists the interval (x_0, x') , $x_0 < x'$, on which $h(x)$ is positive and increasing. This proves (ii).

Since the same results are obtained for $\mu_2 < \mu_1$, the theorem is proved.

When $x_0 \min(\sigma_1, \sigma_2) < |\mu_1 - \mu_2| \leq x_0(\sigma_1 + \sigma_2)$, we can not know whether $G(x;p)$ is unimodal or bimodal. But we shall not be able to know them, so far as $F(x)$ is not known. When $F(x)$ is normal, the results of this theor-

em compared with [1] are not sharp.

We shall now consider the following condition for the shape of the pdf $f(x)$.

Condition 2. There exists the value of x' , $0 < x' < \infty$, such that $f'(x)$ is increasing on the interval $(0, x']$ and nondecreasing on the interval (x', ∞) .

For example, the pdf of the double exponential distribution satisfies this condition.

We can easily obtain the following theorem and the proof is omitted.

Theorem 3. Suppose that Condition 2 is satisfied. If $\mu_1 \neq \mu_2$, then $G(x;p)$ is not unimodal for all p , $0 < p < 1$.

It is immediately obtained by Theorem 3 that under the assumption of Condition 2, $G(x;p)$ is unimodal for all p , $0 < p < 1$, if and only if $\mu_1 = \mu_2$.

3. Examples

As examples of Type A, we shall now give the normal, logistic and t with ν degree of freedom distributions and their pdf's $f(x)$ are given by

$$f(x) = (2\pi)^{-1/2} \exp(-x^2/2), \quad (10)$$

$$f(x) = \exp(-x) [1 + \exp(-x)]^{-2} \quad (11)$$

and

$$f(x) = (\nu^{1/2} B(1/2, \nu/2))^{-1} (1 + x^2/\nu)^{-(\nu+1)/2} \quad (12)$$

respectively.

It is known by [3] that (10), (11) and (12) have the positive points of inflection at $x = 1$, $\log(2 + \sqrt{3})$ and $\sqrt{v/(v+2)}$, respectively. Then, it can be shown by easy but complicated calculations that they satisfy the assumption of Corollary in section 2 by letting x_0 by the above values of inflection, respectively. Therefore, for the normal, logistic and t distributions, we have Corollary and Theorem 2 by letting $x_0 = 1$, $\log(2 + \sqrt{3})$ and $\sqrt{v/(v+2)}$, respectively.

Now the standard deviations of (10), (11) and (12) are 1 , $\pi/\sqrt{3}$ and $\sqrt{v/(v-2)}$ ($v \geq 3$), respectively. Therefore, the standard forms of (11) and (12) have the positive points of inflection at $(\sqrt{3}/\pi)\log(2 + \sqrt{3}) = .73$ and $\sqrt{(v-2)/(v+2)}$, respectively. Thus, by the point of inflection, we may know the tail heaviness of the distribution.

Putting

$$(\sqrt{3}/\pi)\log(2 + \sqrt{3}) = \sqrt{(v-2)/(v+2)},$$

we have $v = 6.7$. Therefore, the logistic distribution can be approximated by the t distribution with 6 or 7 degree of freedom.

For the double exponential distribution of Type B, we have Theorem 3 and note that its pdf has not the point of inflection.

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