



Primeness of Links

Nakanishi, Yasutaka

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Primeness of Links

By Yasutaka NAKANISHI

R. Kirby and W. B. R. Lickorish [7] introduce the method of tangles to prove the primeness of certain knots. In Section 1, we will generalize a notation of tangles and obtain a more applicable machine to links. For examples, we will easily see that some links are Brunnian in Section 2.

We know in [11] that any link is concordant to a prime link with the same Alexander invariant. In Section 3, we will show that any n -components link ($n \geq 2$) is concordant to a prime link with the same Alexander invariant preserving the knot types of components and the double of this concordance is ambient isotopic to the direct product.

F. Hosokawa [5] defines ∇ -polynomials of links and characterizes them as reciprocal polynomials of even degree. In Section 4, we will characterize ∇ -polynomials of ribbon (slice) links in the weak sense as reciprocal polynomials of even degree.

In Section 5, we will elementarily prove *"For each integer n , there are distinct prime knots with the same n -fold cyclic branched covering space."*

This paper should be interpreted in the piecewise-linear category. The standard results and definitions of the link theory is referred to D. Rolfsen's book [12].

1. Prime Links and Tangles

1.1. In this paper, the word *tangle* is borrowed from [3] and is used to mean a pair (B, t) where B is a 3-ball and t is a set of arcs and zero or more loops properly embedded in B , provided that the number of components of t is finite. Two tangles (B_1, t_1) and (B_2, t_2) are said to be *equivalent* if there is a homeomorphism of pairs from (B_1, t_1) to (B_2, t_2) .

1.2. We see examples of tangles in Fig. 1.1; (a) is named *the unknotted tangle*, (b) is *the trivial tangle*, (c) is *the clasp* [7], (d) is *the K-T grabber* [2], and (e) is *the chain*.

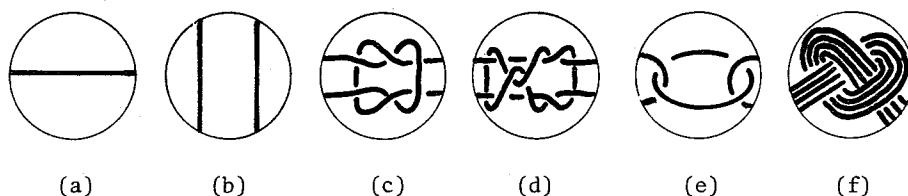


Fig. 1.1

1.3. A tangle (B, t) is *prime* if it has the following properties:

1. (*Locally Trivial*) Any 2-sphere in B , which meets t transversely in two points, bounds in B a ball meeting t in an unknotted spanning arc.
2. (*Inseparable*) The arcs and loops of t cannot be separated by a disc properly embedded in B .
3. (*Indivisible*) Any disc properly embedded in B , which meets t transversely in a single point, divides (B, t) into the unknotted tangle and another.

1.4. Remark. These definitions are slightly different from those of [2], [7], [8] and [11].

1.5. Proposition. *Any locally trivial inseparable tangle, the number of whose arcs is at most two, is a prime tangle.*

Proof. Any properly embedded disc meeting in a single point divides into the unknotted tangle and another. Because the given tangle has at most two arcs, one of tangles divided has only one arc. From locally triviality, it is the unknotted tangle.

1.6. Proposition. *In Fig. 1.1, (b) is not prime, and the others are prime.*

Proof. (a), (b): It is obvious. (c), (d): It is easily checked that they are locally trivial from the unknottedness of each arc. If they were separable, then they must equivalent to the trivial tangle. So they would induce two-bridge knots with ears as in Fig. 1.2. But this is false.

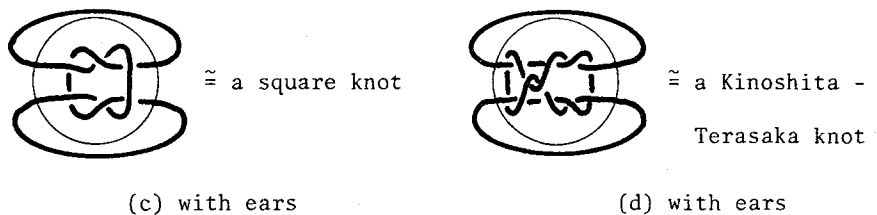


Fig. 1.2

(e): The loop of this tangle is linking arcs, so this tangle is inseparable. Locally triviality is easily checked. (f): For each two arcs, we show the primeness in a ball. If it did not satisfy locally triviality or inseparability, then there would be a ball meeting arcs in a trefoil.

But it induces a trivial knot with ears as in Fig.

1.3. So it is prime.

Each two arcs in a ball is inseparable, so this tangle is indivisible.



$\cong$ a trivial knot

Fig. 1.3

1.7. Remark. On the above argument on (f), "a trefoil" is not essential. Hence parallel arcs on a non-trivial knot in a ball is prime.

1.8. A link L is *prime* if it has the following properties:

1. (*Locally trivial*) Any 2-sphere in S^3 , which meets L transversely in two points, bounds in S^3 a ball meeting L in an unknotted spanning arc.

2. (*Inseparable*) There is no 2-sphere in S^3 that separates the components of L .

1.9. Remark. This definition is that of [4], slightly different from those of [8], [11]. A trivial knot is prime, and a trivial 2-components link is not prime.

1.10. Theorem. Let L be a link in S^3 . Suppose that S^2 is a 2-sphere in S^3 meeting L transversely and separating S^3 into two 3-balls A and B . (1) If $(A, A \cap L)$ and $(B, B \cap L)$ are both inseparable tangles, then L is an inseparable link. (2) If $(A, A \cap L)$ and $(B, B \cap L)$ are both prime tangles, then L is a prime link.

Proof. (1) The proof is contained in the following.

(2) Suppose that F is a 2-sphere in S^3 meeting L transversely in two points; it is required to show that F bounds a ball meeting L in an unknotted arc. The definition of a prime tangle implies that this is true if F is entirely contained in A or in B , so it is in order to proceed by induction on the number of components of $F \cap S^2$.

It may be assumed that F meets S^2 transversely in simple closed curves. Let γ be such a curve innermost on F that bounds a disc δ in F ; it may be assumed without loss of generality that F is in A . This γ may be chosen so that $\delta \cap L$ is empty or a single point. If $\delta \cap L$ is empty, from inseparability, δ cannot separate the components of $A \cap L$ and so there is an isotopy, fixed on L , which reduces the number of components of $F \cap S^2$. On the other hand, if $\delta \cap F$ is a single point, from indivisibility, δ divides $(A, A \cap L)$ into the unknotted tangle and another. Then there is an isotopy that reduces the number of components of $F \cap S^2$ and which keeps L set-wise fixed. Hence by induction, F bounds a ball meeting L in an unknotted arc.

Finally, if F is a 2-sphere in S^3 disjoint from L , then by similar argument as above, F may be isotoped into A keeping L fixed. Thus F bounds a ball disjoint from L . So L is a prime link.

The proof is complete.

1.11. Remark. A prime link may not be always divided into two prime tangles. But a prime link can be divided into one trivial tangle and one prime tangle (see Lemma 3.1).

1.12. Theorem. Let (C, v) be a tangle and let D be a disc properly embedded in C that separates (C, v) into two tangles (A, t) and (B, u) and that the number of points of $(\partial A - D) \cap v$, $(\partial B - D) \cap v$, $D \cap v$ is greater than one respectively. And let (B, u) be an inseparable tangle.

(1) For any disc Δ properly embedded in A with $\Delta \cap \partial D = \emptyset$, Δ does not separate t in A if $\Delta \cap t = \emptyset$. Then (C, v) is inseparable.

(2) On the condition continued, for any disc Δ properly embedded in A with $\Delta \cap \partial D = \emptyset$, Δ does not divide (A, t) into the unknotted tangle and another if Δ meets t in a single point. And if (B, u) is a prime tangle and (A, t) is a locally trivial tangle, then (C, v) is a prime tangle.

Proof. (1) The proof is essentially contained in the following.

(2) Suppose that F is a 2-sphere in C meeting v transversely in two points. Then, by similar argument to Proof of Theorem 1.10, F may be isotoped into A or into B keeping v fixed. From locally triviality, F bounds a ball meeting v in an unknotted arc.

Suppose that Δ is a disc properly embedded in C , which is disjoint from v or meets v transversely in a single point, it is required to show that Δ does not separate v in C or does not divide (C, v) into the unknotted tangle and another.

First, we consider whether the number of components of $\Delta \cap D$ can be reduced or not. It may be assumed that Δ meets D transversely in simple closed curves and proper arcs. By similar argument to Proof of Theorem 1.10, simple closed curves are reducible. So it may be assumed that $\Delta \cap D$ contains only proper arcs. Let γ be such a curve innermost on Δ i.e.

γ separates Δ into two discs, one of which, say δ , satisfies $\delta \cap D = \gamma$. This γ may be chosen so that $\delta \cap v$ is empty. If δ is in B , from inseparability of (B, u) , δ cannot separate the components of u and so there is an isotopy, fixed on v , which reduces the components of $\Delta \cap D$.

It may be assumed that δ is in A . If δ does not separate t in A , there is an isotopy, fixed on v , which reduces the components of $\Delta \cap D$. It may be assumed that δ separates t in A .

Consider a collection of such discs δ_i 's properly embedded in A that $\delta_i \cap D$ has only one component for each i and $\delta_i \cap \delta_j = \emptyset$ for each $i \neq j$ and $\delta_i \cap t = \emptyset$ for each i . Since the number of components of t is finite, there are at most finite number of these discs essentially.

Let $\gamma_i = \delta_i \cap D$ and let γ_1 is innermost on D .

If δ_1 does not separate t in A , it is not essential.

δ_1 separates (A, t) into two tangles (A_1, t_1) and (A_2, t_2) .

If they are equivalent to the unknotted tangle, then (C, v) itself is equivalent to (B, u) , so is a prime tangle.

If only one of them, say (A_1, t_1) , is equivalent to the unknotted tangle, then $(A_1 \cup B, t_1 \cup u)$ is equivalent to (B, u) and so prime.

If they are not equivalent to the unknotted tangle, we assume that γ_1 bounds an innermost disc, $A_1 \cap D$, on D . It is easily checked that $(A_1 \cup B, t_1 \cup u)$ is a prime tangle.

On the last two case, we obtain a new disc D' dividing (C, v) into a prime tangle and another, which satisfies the condition of this Theorem and the number of points of $D' \cap v$ is less than that of $D \cap v$.

Hence, by induction, (C, v) is shown to be a prime tangle.

The proof is complete.

1.13. A link L in $S^1 \times B^2$ is *locally trivial* if any 2-sphere in $S^1 \times B^2$, which meets L transversely in two points, bounds in $S^1 \times B^2$ a ball meeting L in an unknotted spanning arc. The *wrapping number* of L is the minimum number of intersections of L with a meridian disc of $S^1 \times B^2$. If L is a link in $S^1 \times B^2$ and K is a knot in S^3 , the L *cable* of K is a link in S^3 formed by mapping $S^1 \times B^2$ into $N(K)$, and considering the image of L under this map (cf. [9]).

1.14. Theorem. (Cf. [9, Theorem 4.2]) *Let L be a link in $S^1 \times B^2$ with wrapping number > 1 , and let K be any non-trivial knot in S^3 . The L cable of K is prime if and only if L is locally trivial in $S^1 \times B^2$.*

The proof is parallel to the argument in [9], so we omit it.

1.15. Remark. On the above, if K is not prime, the L cable of K has a decomposition into prime tangles from 1.7 and 1.12.

2. Prime Links and Brunnian Property

2.1. If a link is non-trivial, yet every proper sublink is trivial, we say that it has the *Brunnian property* or that it is a *Brunnian link*.

2.2. A Brunnian link is easily shown to be a prime link. Conversely, if a link, whose proper sublinks are all trivial links, is inseparable, it is Brunnian link. On the following examples, it is easily checked that every proper sublink is trivial. So, by showing it inseparable (prime), we know them to be Brunnian links.

2.3. The *Borrromean rings*, as in Fig. 2.1, is a well-known Brunnian

link. It has a decomposition into prime tangles; the chain (see 1.6), as in Fig. 2.1. From Theorem 1.10, it is prime, and so is a Brunnian link.

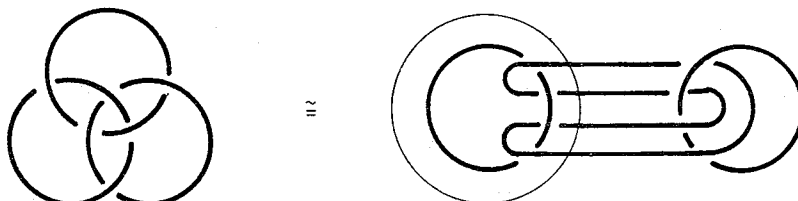


Fig. 2.1

2.4. On the above, each component is homotopically linked from the remaining components. The followings are not so.

2.5. Consider the links ${}_1L_n$ and ${}_2L_n$ as in Fig. 2.2 where the subscripts n means the number of components. They have a decomposition into prime tangles; the clasp or the K-T grabber (see 1.6), as in Fig. 2.2 respectively. From Theorems 1.10 and 1.12, they are prime links, and so are Brunnian links.

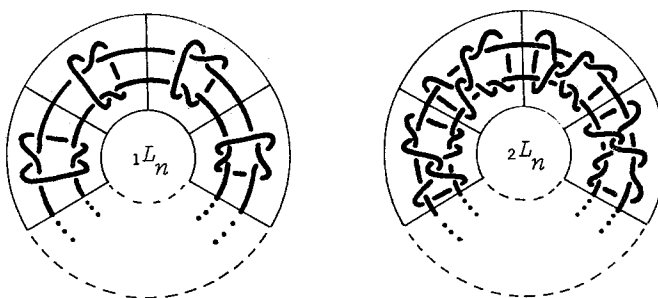


Fig. 2.2

2.6. Consider the link ${}_3L_n$ as in Fig. 2.3, where the subscript n means the number of components. ${}_3L_2$ is equivalent to ${}_1L_2$. Here we show that ${}_3L_n$ ($n \geq 4$) has a decomposition into prime tangles.

Consider the tangle (i) in Fig. 2.4. If an arc was separable from the remainder, then the remainder itself was completely separable, and so the tangle itself was completely separable. Hence it would induce a link separated from the loop with ears as in Fig. 2.5. But this is false.

Since the tangle (i) without the loop is completely separable and each arc is unknotted, locally triviality is easily seen.

Since the tangle (i) without one component is completely separable, two tangles divided by a disc would be completely separable, if a disc could divide the tangle (i) into two tangles which are not the unknotted tangle. So the tangle (i) itself was completely separable. But this is false.

Hence the tangle (i) is a prime tangle.

The tangle (ii) in Fig. 2.4 is easily checked to satisfy the condition of (A, t) in Theorem 1.12. So, the sum of (i) and (ii) is also a prime tangle, from Theorem 1.12.

Hence we know that ${}_3L_n$ ($n \geq 4$) has a decomposition into prime tangles.

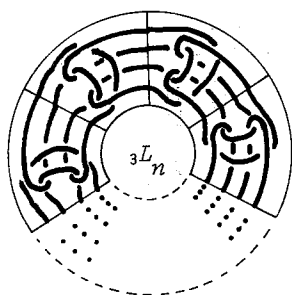
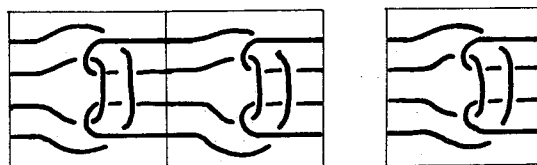


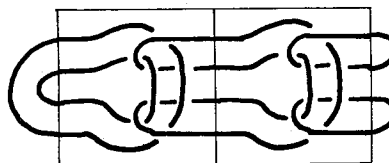
Fig. 2.3



(i)

(ii)

Fig. 2.4



(i) with ears

Fig. 2.5

3. Primeness and Alexander Invariants of Concordant Links

3.1. Lemma. (Cf. [2], [7], [8]) *Let L be a link in S^3 . Then there exists, embedded in S^3 , a 2-sphere meeting L transversely in four points and separating S^3 into two 3-balls A and B such that*

- (i) $(A, A \cap L)$ *is equivalent to the trivial tangle,*
- (ii) $(B, B \cap L)$ *is a prime tangle.*

Proof. We know the existence and uniqueness of prime decomposition of inseparable links by Y. Hashizume [4]. Hence there is a locally trivial decomposition of a link without uniqueness. First, we show the above for locally trivial links. Let denote by $n(L)$ the number of components of a link L .

On the case $n(L) = 1$, a locally trivial knot is a prime knot. If it is a trivial knot, we take a 2-sphere as in Fig. 3.1. If it is not a trivial knot, there is two arcs whose meridians are not homotopic in the exterior of the knot. We take a 2-sphere that bounds the trivial tangle whose arcs are just the above arcs. It easily checked that the exterior of this tangle is a locally trivial tangle, let denote it (B', t') . We illustrate it as in Fig. 3.2. It may be deformed into Fig. 3.3. Since they are easily checked the condition of Theorem 1.12, (B, t) in Fig. 3.3 is prime.



Fig. 3.1

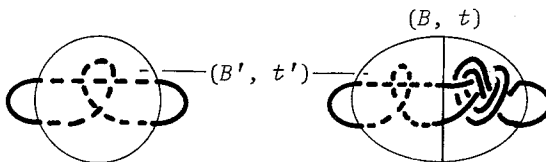


Fig. 3.2

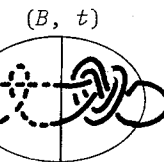


Fig. 3.3

On the case $n(L) = 2$, a locally trivial 2-components link which is not prime is only a trivial 2-components link. If L is a trivial 2-components link, we take a 2-sphere as in Fig. 3.4. If L is not so, it is a prime link. If L has a knotted components, we take a 3-ball A in S^3 such that $(A, A \cap L)$ is equivalent to the trivial tangle and that A contains a subarc of each component of L . Let B be the exterior of A . We will show that $(B, B \cap L)$ is a prime tangle. Locally triviality is immediate as L is locally trivial. $B \cap L$ contains a knotted arc from the assumption. This fact and locally triviality imply inseparability. From Proposition 1.5, $(B, B \cap L)$ is a prime tangle. If L has no knotted components, we take a 3-ball as similar to the above. $(B, B \cap L)$ is easily checked to be a locally trivial tangle. We illustrate it as in Fig. 3.5. It may be deformed into Fig. 3.6. Since they are easily checked with the condition of Theorem 1.12, (B^*, t^*) in Fig. 3.6 is a prime tangle.

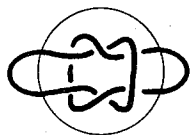


Fig. 3.4

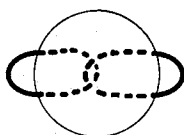


Fig. 3.5

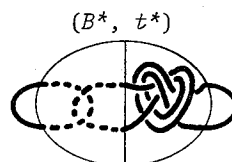


Fig. 3.6

On the case $n(L) \geq 3$, a locally trivial link is a prime link. We take a 3-ball as similar to the above. We will show that $(B, B \cap L)$ is a prime tangle. Locally triviality is immediate as L is locally trivial. If there was a proper disc in B separating $B \cap L$ in B . From locally triviality, it separated arcs. But consider the part containing loops, it is easily seen that this part is not locally trivial. So $(B, B \cap L)$ is inseparable and so is prime.

For each locally trivial factor of a given link, we can construct a prime tangle as above, and a sum of these prime tangles with the trivial tangle reproduces a given link in S^3 . From Theorem 1.12, this sum is a prime tangle, too. The proof is complete.

3.2. Lemma. ([11, Lemma 2]) *Let $L' = L \cup k$ be a link, where k is a trivial knot. Suppose that k has linking number 0 with each component of L and that k bounds a ribbon disc in $S^3 - L$. We denote by L^* a link which is obtained from L by a surgery along k in S^3 . Then both L and L^* have the same Alexander invariants.*

3.3. Lemma. *The substitution the K-T grabber for the trivial tangle on a regular projection of a link does not change the concordance class and the Alexander invariant.*

Proof. From a surgical description of the K-T grabber as in Fig. 3.7, which is deformed to Fig. 3.8, we see this surgery curve bounds a ribbon disc in the complement of the tangle in the 3-ball. From Lemma 3.2, this substitution does not change the Alexander invariant. Ribbon move at β in Fig. 3.7 shows that the K-T grabber is concordant to the trivial tangle.



Fig. 3.7

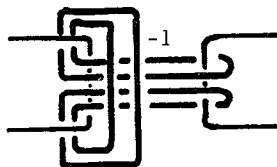


Fig. 3.8

3.4. Lemma. *The double of the above concordance is ambient isotopic to the direct product.*

Proof. The K-T grabber and the band β in Fig. 3.7 can be regarded as in Fig. 3.9. The above concordance is a ribbon concordance obtained by a fission along the band β . We obtain β' from β by interchanging over- and under- crossings of the band as in Fig. 3.10. The double of concordance obtained by a fission along β and that along β' are ambient isotopic (cf. [15, Proposition 2.15]). Fig. 3.10 is deformed to Fig. 3.11. It is easily checked that the double is ambient isotopic to the direct product. This completes the proof.

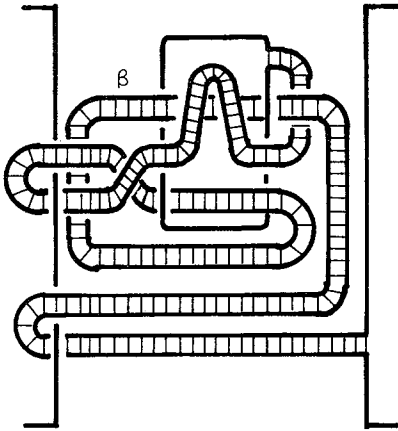


Fig. 3.9

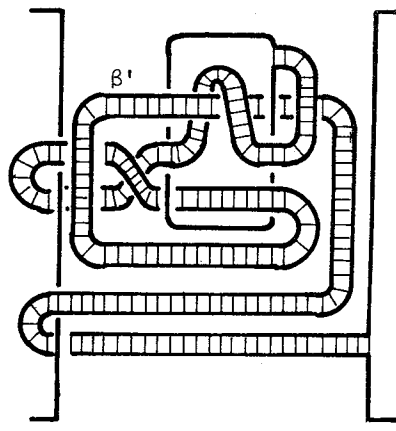


Fig. 3.10

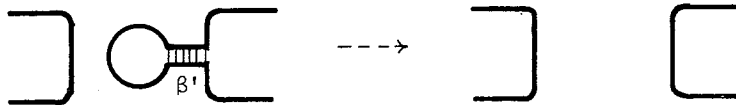


Fig. 3.11

3.5. Theorem. *Any n -components link is concordant to a prime link with the same Alexander invariant and the double of this concordance is ambient isotopic to the direct product. Furthermore, if n is greater than one, this concordance can be assumed to preserve the knot types of components of a link.*

Proof. By Lemma 3.1, we have a tangle decomposition of a link in S^3 . From Lemma 3.3, we substitute the K-T grabber for $(A, A \cap L)$, and obtain a new link which is concordant to a given one and has the same Alexander invariant. From Lemma 3.4, the double of this concordance is ambient isotopic to the direct product. From Theorem 1.10, this link is prime. It is easily checked that the knot types of components are preserved if $n \geq 2$. Because each arc of the K-T grabber is unknotted. The proof is complete.

3.6. On the case $n = 1$; "Any knot is concordant to a prime knot with the same Alexander polynomial" is proved by S. A. Bleiler [2].

3.7. Corollary. *Any set of surfaces embedded in a 4-space has the cross-section which is a prime link.*

3.8. Corollary. (With [16, Theorem 2.8]) *Any ribbon 2-knot has a prime equatorial knot.*

Proof. T. Yanagawa [16] proves "Any ribbon n -knot ($n \geq 2$) has an equatorial knot." Apply the argument in this section. Since the above concordance preserves ribbon property, we obtain a required one.

3.9. Corollary. *For any knot K , there is a prime knot K' with the same Alexander invariant such that $K \# K'$ is doubly null-cobordant.*

Proof. It is well-known that $K \# K^*$ is doubly null-cobordant where K^* means the reflection of K . Apply the argument in this section to K^* , and we obtain a required one.

3.10. Note. There is another construction to show Theorem 3.5 without preserving the knot types of components as follows.

It is known that any link is represented as a closed braid. Its wrapping number may be assumed to be greater than one. Let denote it by n . Consider the $(n, 0)$ -cable link of a Kinoshita-Terasaka knot. We obtain a new link from the above closed braid and the above $(n, 0)$ -cable link by natural complete fusion as illustrated in Fig. 3.12. On the analogy of Lemmas 3.3 and 3.4, we know this new link to be a required one by Theorem 1.14.

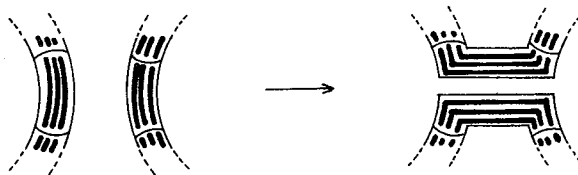


Fig. 3.12

4. ∇ -Polynomials of Ribbon Links in the Weak Sense

4.1. A link is a *ribbon link in the weak sense* iff it bounds an immersed oriented surface of genus 0 whose singularities are all ribbon-types, or equivalently, iff it introduces a ribbon knot by only fusions.

The ∇ -polynomial $\nabla(t)$ of a link is $\Delta(t_1, \dots, t_n) / (t-1)^{n-2}$ where $\Delta(t_1, \dots, t_n)$ is the Alexander polynomial of a given link and n is the number of components.

4.2. Theorem. For any reciprocal polynomial of even degree $f(t)$, there exists an n -components prime ribbon link in the weak sense whose ∇ -polynomial is equal to $f(t)$ for an arbitrarily integer $n \geq 2$.

Proof. Adding the following condition (*) to Hosokawa's construction in [5], we obtain a ribbon link in the weak sense.

(*) $a_2 \cup a_4 \cup \dots \cup a_{2h}$ is a trivial link.

Because we can obtain a ribbon knot from the link by fusions along B_{2h+i} 's ($1 \leq i \leq n-1$).

We can construct a prime link from the above link as in Section 3. Since ribbon concordance preserves ribbon property in the weak sense, this link is a required one. This completes the proof.

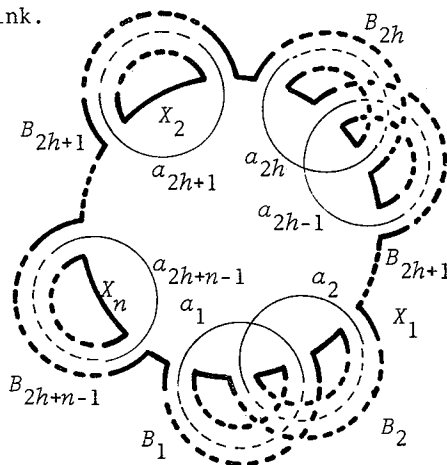


Fig. 4.1

4.3. Corollary. ∇ -polynomials of ribbon links in the weak sense are characterized as reciprocal polynomials of even degree. Those of slice links in the weak sense are also characterized as the same.

Proof. F. Hosokawa characterizes ∇ -polynomials of links as reciprocal polynomials of even degree in [5]. A ribbon link in the weak sense is one of slice links in the weak sense, and is one of links. From Theorem 4.2, those ∇ -polynomials are characterized as the above.

5. Prime Knots with the Same Cyclic Branched Covering Spaces

5.1. Preliminaries. Let $L = J \cup K$ be a 2-components link such that J and K are trivial knots with $lk(J, K) = 1$. Let denote by $\Sigma_n(K)$ (resp. $\Sigma_n(J)$) the n -fold cyclic branched covering space of K (resp. J), and let denote by J_n (resp. K_n) the lift of J (resp. K) in $\Sigma_n(K)$ (resp. $\Sigma_n(J)$). Since K and J are trivial, $\Sigma_n(K)$ and $\Sigma_n(J)$ are homeomorphic to S^3 , J_n and K_n are also knots in S^3 .

The n -fold cyclic branched covering spaces of J_n and K_n are homeomorphic to the $\mathbb{Z}_n \oplus \mathbb{Z}_n$ -covering space of L , and so J_n and K_n have the homeomorphic n -fold cyclic branched covering spaces.

If $L = J \cup K$ is not interchangeable i.e. there is no homeomorphism of S^3 mapping J to K , then J_n and K_n may have distinct knot types.

5.2. Examples. ([10], [13]) Consider the link 9^2_{34} , 9^2_{35} , or 9^2_{42} in the table of [12, Appendix C]. Let denote by $L = J \cup K$ one of them. J_n and K_n are known to have distinct knot types from the fact: their polynomials are distinct (see, for a example, M. Sakuma [14]).

5.3. Primeness of J_n and K_n on the case 9^2_{34} . 9^2_{34} is illustrated in Fig. 5.1. Let denote by $\tilde{J}_i$ (resp. $\tilde{K}_i$) an i -tuple of a fundamental region of the infinite cyclic covering space of K (resp. J). We consider them as tangles. $\tilde{J}_1$, $\tilde{J}_3$, $\tilde{K}_1$, and $\tilde{K}_3$ is illustrated in Fig. 5.2.

First we show that $\tilde{J}_3$ is a prime tangle. $\tilde{J}_3$ is considered as a sum of the tangles (i) and (ii) in Fig. 5.3. We denote the tangle (i) by $(B, t_1 \cup t_2 \cup t_3)$ as in Fig. 5.3. The arc t_1 has knotted prime factors 3_1 and 6_3 in B . The arcs t_2 and t_3 are unknotted in B .

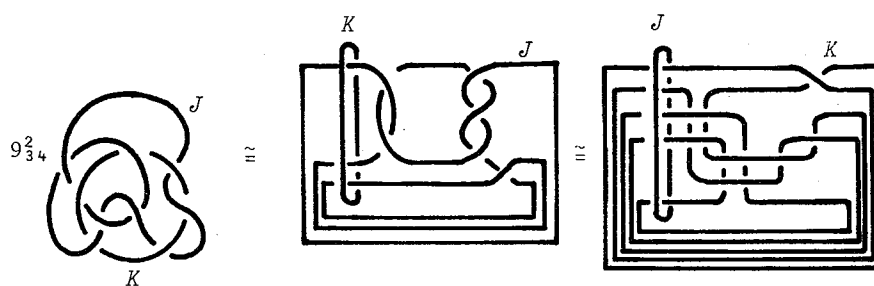


Fig. 5.1

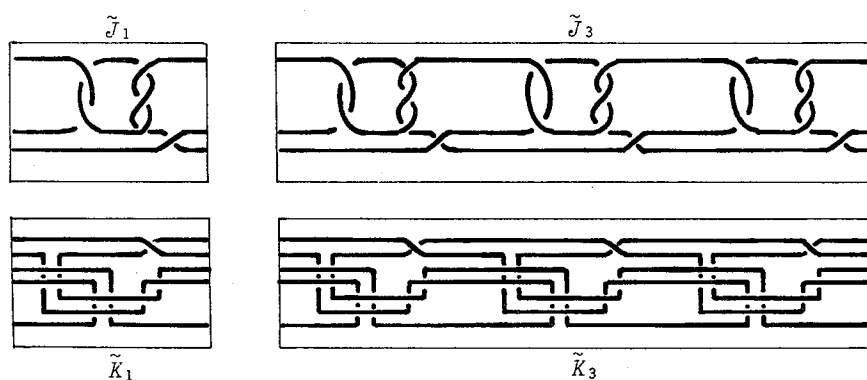


Fig. 5.2

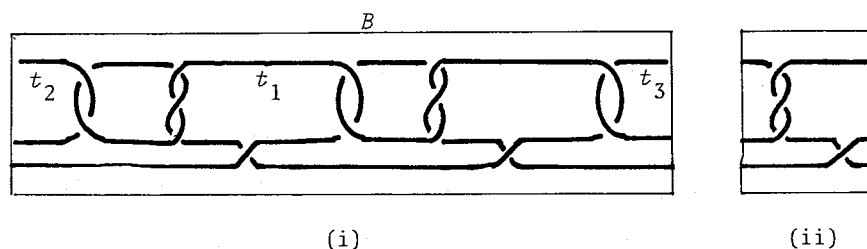


Fig. 5.3

5.4. Lemma. Let $(B, t_1 \cup t_2)$ be a tangle such that at least one of t_1 and t_2 is non-trivial in B . If $(B, t_1 \cup t_2)$ with ears is a prime knot and has no knotted factor of t_1 nor t_2 , then this tangle is prime. Here (B, t) with ears means an union of (B, t) and the trivial tangle.

Proof. If $(B, t_1 \cup t_2)$ does not satisfy locally triviality or inseparability, then $(B, t_1 \cup t_2)$ with ears must have at least one knotted factor of t_1 or t_2 from non-triviality of t_1 or t_2 . This contradicts the assumption. Hence $(B, t_1 \cup t_2)$ is a prime tangle.

5.5. Lemma. Let $(B, t_1 \cup t_2)$ be a tangle such that at least one of t_1 and t_2 is non-trivial in B . If $(B, t_1 \cup t_2)$ with ears does not have all knotted factors of t_1 and t_2 , then this tangle is inseparable.

The proof is obvious, so we omit it.

5.6. Lemma. Let $(B, t_1 \cup t_2 \cup t_3 \cup \dots \cup t_n)$ be a tangle such that t_1 is a non-trivial arc and t_i 's ($2 \leq i \leq n$) are unknotted arcs. If $(B, t_1 \cup t_2)$ is a prime tangle and if $(B, t_1 \cup t_i)$ ($3 \leq i \leq n$) are inseparable tangles and at least one of knotted factors of t_1 is locally trivial in $(B, t_1 \cup t_i)$ respectively for each i , then this tangle is a prime tangle.

Proof. Locally triviality and inseparability is immediate.

If a disc D meeting t in a single point divides $(B, t = t_1 \cup \dots \cup t_n)$ into two tangles which are not the unknotted tangles, then D must meet only with t_1 from inseparability of $t_1 \cup t_i$ ($2 \leq i \leq n$) in B . We denote two tangles divided by B_1 and B_2 such that B_2 contains t_2 . By the assumption, there is t_j ($3 \leq j \leq n$) in B_1 . Since $(B, t_1 \cup t_2)$ is prime, $(B_1, t_1 \cap B_1)$ is the unknotted tangle. Then all knotted factors of t_1 are not locally trivial in $(B, t_1 \cup t_j)$. This contradicts the assumption. Hence $(B, t_1 \cup \dots \cup t_n)$ is a prime tangle.

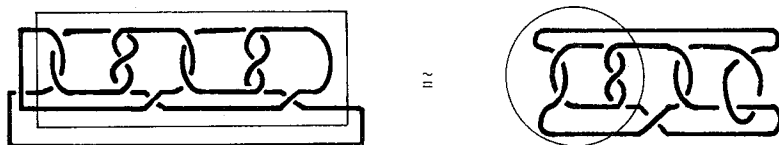
5.7. Proposition. *The tangle in Fig. 5.4 is a prime tangle.*



Fig. 5.4

Proof. It is easily checked that there is a union of this tangle and the trivial tangle which is a trivial knot. From Lemma 5.4, this tangle is a prime tangle.

5.8. Continued from 5.3. $(B, t_1 \cup t_2)$ with ears as in Fig. 5.5 is a prime knot from 1.10 and 5.7, and has no knotted factor of t_1 . From 5.4, this tangle is a prime tangle. $(B, t_1 \cup t_3)$ with ears as in Fig. 5.6 is a knot which does not have a knotted factor of t_1 . From 5.5, this tangle is an inseparable tangle. From 5.6, the tangle (i) is prime tangle.



$11_4 4_3 \quad (21, 2+)(3, 2-)$ in [3] with $\Delta(t) = t^6 - 3t^5 + 5t^4 - 5t^3 + 5t^2 - 3t + 1$

Fig. 5.5

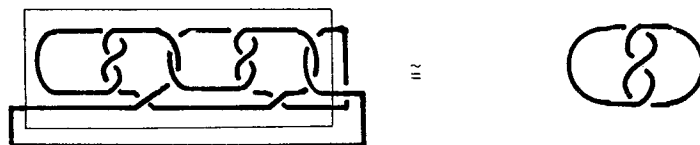


Fig. 5.6

$\tilde{J}_3$: a sum of the tangles (i) and (ii) in Fig. 5.3 is easily checked to satisfy the condition of Theorem 1.12, and so is a prime tangle.

$\tilde{J}_4$: a sum of $\tilde{J}_1$ and $\tilde{J}_3$ is easily checked to satisfy the condition of Theorem 1.12, and so is a prime tangle. On a similar way, we can show

that $\tilde{J}_n$ ($n \geq 3$) is a prime tangle.

Since the knot J_n may be considered as a union of $\tilde{J}_i$ and $\tilde{J}_j$ for $i+j=n$, J_n ($n \geq 6$) are prime knots from Theorem 1.10.

J_2 , J_3 , J_4 , and J_5 have a decomposition into prime tangles as in Fig. 5.7. Hence they are prime knots from Theorem 1.10.

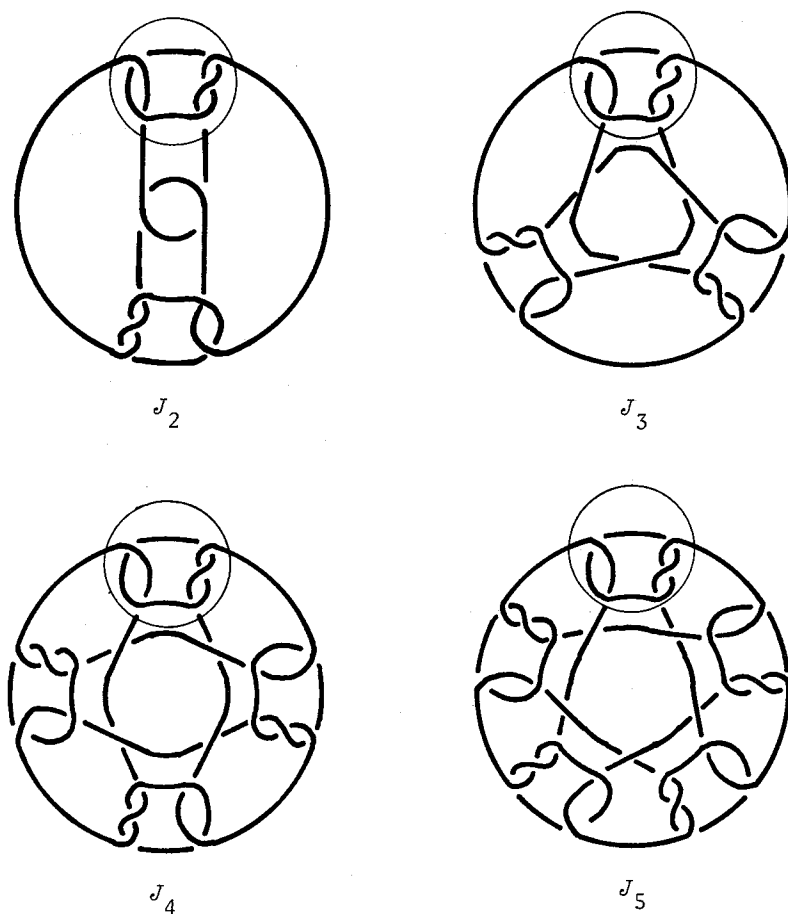


Fig. 5.7

Next we show that $\tilde{K}_3$ is a prime tangle. We denote $\tilde{K}_3$ by $(B, t = t_1 \cup t_2 \cup t_3 \cup t_4 \cup t_5)$ as in Fig. 5.8. t_1 is 4_1 and t_i ($i=2,3,4,5$) is unknotted arc in B . $(B, t_1 \cup t_i)$ ($i=2,3,4,5$) with ears as in Fig. 5.9 have no knotted factor of t_1 and are prime knots. From Lemma 5.4, these tangles are prime tangles. From Lemma 5.6, (B, t) is a prime tangle.

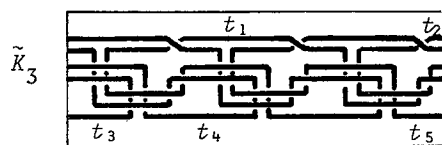


Fig. 5.8

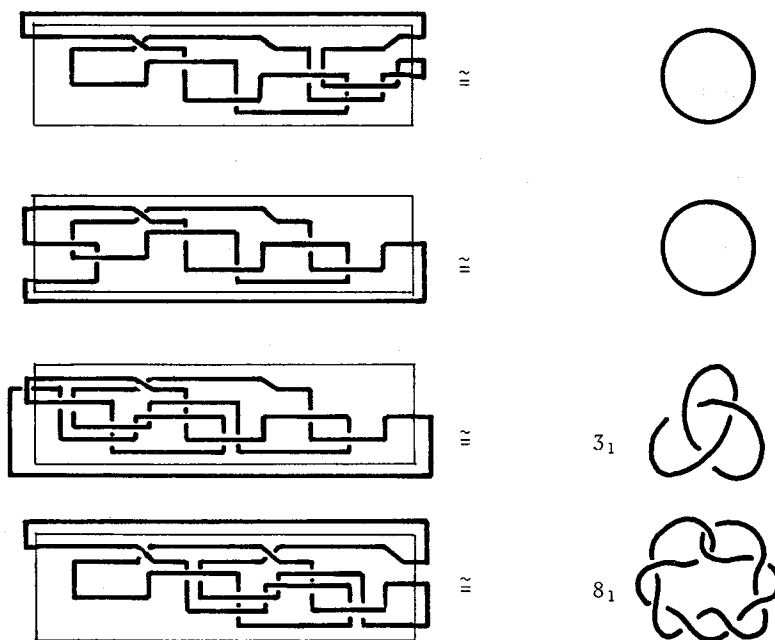


Fig. 5.9

$\tilde{K}_4$: a sum of $\tilde{K}_1$ and $\tilde{K}_3$ is easily checked to satisfy the condition of Theorem 1.12, and so is a prime tangle. On a similar way, we can show that $\tilde{K}_n$ ($n \geq 3$) is a prime tangle.

Since the knot K_n may be considered as a union of $\tilde{K}_i$ and $\tilde{K}_j$ for $i+j=n$, K_n ($n \geq 6$) are prime knots from Theorem 1.10.

K_2 is a doubled knot of a trefoil knot with 6-twists as illustrated in Fig. 5.10, and so is a prime knot from Theorem 1.14.



Fig. 5.10

The author does not know whether K_3 , K_4 , and K_5 have a decomposition into prime tangles or not. So the proof of their primeness is complicated, and we omit it here.

5.9. On the case 9_{35}^2 : 9_{35}^2 is illustrated in Fig. 5.11.

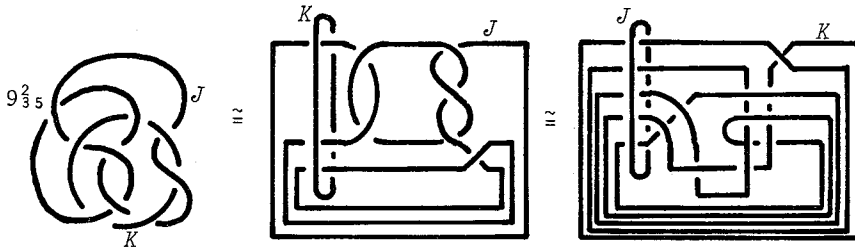


Fig. 5.11

On a similar way as 9_{34}^2 , $\tilde{J}_n$ and $\tilde{K}_n$ ($n \geq 3$) are prime tangles. Hence J_n and K_n ($n \geq 6$) have a decomposition into prime tangles. Also J_2 , J_3 , J_4 , and J_5 have a decomposition into prime tangles. K_2 is a doubled knot of a trefoil knot with -6-twists.

5.10. On the case 9_{42}^2 . 9_{42}^2 is illustrated in Fig. 5.12.

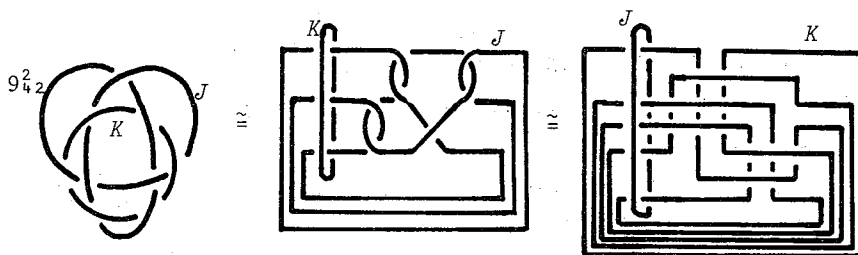


Fig. 5.12

On a similar way as $9^2_{3,4}$, $\tilde{J}_n$ and $\tilde{K}_{n+2}$ ($n \geq 1$) are prime tangles.

Hence J_n and K_{n+4} ($n \geq 2$) have a decomposition into prime tangles.

5.11. Theorem. *For each integer n , there exists distinct prime knots whose n -fold cyclic branched covering spaces are homeomorphic.*

The argument in this section proves the above except $n = 3, 4, 5$.

5.12. Remark. M. Sakuma says the same result as Theorem 5.11 in [14].

5.13. Question. Are the followings equivalent?

- (1) $L = J \cup K$ is not interchangeable.
- (2) $(\Sigma_n(K), J_n)$ is not homeomorphic to $(\Sigma_n(J), K_n)$ for all n .
- (3) $(\Sigma_n(K), J_n)$ is not homeomorphic to $(\Sigma_n(J), K_n)$ for some n .

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Department of Mathematics
 Kobe University
 Nada-ku, Kobe, 657
 Japan

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