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Non-Archimedean Quasi 2-normed Spaces

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The theory of quasi-normed spaces and non-archimedean quasi-normed spaces has been studied by T. Konda in a series of papers. While quasi 2-normed spaces has been discussed by A. H. Siddiqi and J. Esfahamizadeh. The aim of the present paper is to introduce and study the concept of non-archimedean quasi 2-normed spaces.

Definition. Let X be a vector space of dimension greater than 1 over field K with non-trivial non-archimedean valuation. A real valued function $u(x, y)$ over $X \times X$ is called a *non-archimedean quasi 2-norm with power p* , $0 < p \leq 1$, if it satisfies the following conditions.

- (i) $u(x, y) = 0$ if and only if x and y are linearly dependent,
 - (ii) $u(x, \beta y) = |\beta|^p u(x, y)$ for $\beta \in K$,
 - (iii) $u(x, y) = u(y, x)$ for all $x, y \in X$,
 - (iv) $u(x, y + z) \leq \max(u(x, y), u(x, z))$ for all $x, y, z \in X$.
- $(X, u(\cdot, \cdot))$ is called a *non-archimedean quasi 2-normed space*.

Theorem 1. The condition (iv) is equivalent to the following.

$$(iv)' \quad u(x + z, y + z) \leq \max(u(x, y), u(y, z), u(z, x)).$$

That is, a non-archimedean quasi 2-norm satisfies (iv)'.

Proof. Suppose u is a non-archimedean quasi 2-norm, then by (iv) of the definition, we get

$$\begin{aligned} u(x+z, y+z) &\leq \max(u(x+z, y), u(x+z, z)) \\ &\leq \max(u(x, y), u(z, y), u(x, z), u(z, z)) . \end{aligned}$$

Hence

$$u(x+z, y+z) \leq \max(u(x, y), u(z, y), u(x, z))$$

because $u(z, z) = 0$. Now, we suppose that (i), (ii), (iii) and (iv) hold. Then

$$\begin{aligned} u(x, y+z) &= u(-\alpha z + (x + \alpha z), (y+z-x-\alpha z) + (x + \alpha z)) \\ &\leq \max(u(-\alpha z, y+z-x-\alpha z), u(y+z-x-\alpha z, x+\alpha z), \\ &\quad (x+\alpha z, -\alpha z)) \\ &\leq \max(u(x+\alpha z, y+z), |\alpha|^p(u(y-x, z), u(x, z))). \end{aligned}$$

Since

$$u(x+\alpha z, y+\alpha z) \leq \max(u(x, \alpha y), u(\alpha y, \alpha z), u(\alpha z, x))$$

by (iv)', we get

$$|\alpha|^p u(x+z, y+z) \leq \max(|\alpha|^p u(x, y), |\alpha|^{2p} u(y, x), |\alpha|^p u(z, x))$$

by (i). Or

$$u(x+\alpha z, y+z) \leq \max(u(x, y), |\alpha|^p u(y, z), u(x, z)) \text{ for } \alpha \neq 0 .$$

Thus

$$\begin{aligned} u(x, y+z) &\leq \max(u(x, y), u(x, z), |\alpha|^p u(y, z), \\ &\quad \{u(y-x, z), u(x, z)\}) . \end{aligned}$$

By taking $\alpha \rightarrow 0$, we have $u(x, y+z) \leq \max(u(x, y), u(x, z))$. This completes the proof of the theorem.

Theorem 2. Let X be a non-archimedean quasi 2-normed space and α be any real number, then $u(x, y) = u(x, y + \alpha x)$.

Proof. For any α and any points $x, y \in X$, we have

$$\begin{aligned} v(x, y) &= v(x, y + \alpha x - \alpha x) \\ &\leq \max(v(x, y + \alpha x), v(x, -\alpha x)) \quad (\text{by (iv)}) \\ &= v(x, y + \alpha x) \quad \text{as } v(x, -\alpha x) = 0 \quad (\text{by (i)}) \end{aligned}$$

and

$$v(x, y + \alpha x) \leq \max(v(x, y), v(x, \alpha x)) = v(x, y).$$

Therefore $v(x, y) = v(x, y + \alpha x)$.

Theorem 3. Every non-archimedean quasi 2-normed space X is a locally K -convex topological vector space.

Proof. Let U be a given neighborhood of $a - b$. By Theorem 2.1 in [1], there is a neighborhood in U of $a - b$ of the form $W_{\Sigma}(a-b)$, $\Sigma = \{(b_1, \varepsilon_1), (b_2, \varepsilon_2), \dots, (b_n, \varepsilon_n)\}$. $\Sigma_b = \{(a-b_1, \varepsilon_1), (a-b_2, \varepsilon_2), \dots, (a-b_n, \varepsilon_n)\}$. For $i = 1, 2, \dots, n$ and $a' \in W_{\Sigma_a}(a)$, where $\Sigma_a = \{(b_1 + b, \varepsilon_1), (b_2 + b, \varepsilon_2), \dots, (b_n + b, \varepsilon_n)\}$, $v(b_i - a + b, a' - a) = \sigma(a, b_i + b, a') < \varepsilon_i$, where σ is the ultra 2-metric induced by non-archimedean quasi 2-norm. For every $b' \in W_{\Sigma_b}(b)$, $v(b_i - a + b, b' - b) = v(-b_i + a - b, b' - b) = \sigma(b, a_i - b, b') < \varepsilon_i$, so we have

$$\begin{aligned} \sigma(a - b, b_i, a' - b') &= v(b_i - a + b, a' - b' - a + b) \\ &\leq \max(v(b_i - a + b, a' - a), v(b_i - a + b', b' - b)) \\ &\leq \max(\varepsilon_i, \varepsilon_i) = \varepsilon_i. \end{aligned}$$

That is for every point $a' \in W_{\Sigma_a}(a)$ and every point $b' \in W_{\Sigma_b}(b)$, $a' - b' \in W_{\Sigma}(a-b) \subseteq U$. Therefore the mapping $(a, b) \rightarrow a - b$ over $X \times X$ into X is continuous.

Now we show that the mapping $(\alpha, a) \rightarrow \alpha a$ of $R \times X$ into X is continuous. Let U be a neighborhood of a for $\alpha \in R$ and $a \in X$. By Theorem 2.1 in [1] there is in U a neighborhood of a of the form

$W_{\Sigma}(\alpha a)$ with $\Sigma = \{(b_1, \varepsilon_1), (b_2, \varepsilon_2), \dots, (b_n, \varepsilon_n)\}$. Let

$$m_i = \begin{cases} \frac{\varepsilon_i}{v(a, b_i)} & \text{for } (a, b_i) \neq 0 \\ 1 & \text{otherwise} \end{cases}, \quad m = \inf m_i, \quad \varepsilon_{i'} = \frac{\varepsilon_i}{\{|\alpha|^p + m\}}$$

and $\Sigma' = \{(b_1 + a - \alpha a, \varepsilon_{1'}), (b_2 + a - \alpha a, \varepsilon_{2'}), \dots, (b_n + a - \alpha a, \varepsilon_{n'})\}$.

Then for every real number with $|\alpha - \alpha'|^p < m$ and every point $a' \in W_{\Sigma}(a)$,

$$\begin{aligned} \sigma(\alpha a, b_i, \alpha' a') &= v(b_i - \alpha a, \alpha' a' - \alpha a) \\ &= v(b_i - \alpha a, (\alpha' a' - \alpha' a) + (\alpha' a - \alpha a)) \\ &\leq \max(v(b_i - \alpha a, (\alpha' a' - \alpha' a)), v(b_i - \alpha a, (\alpha' a - \alpha a))) \\ &\leq \max(|\alpha'|^p (a, b_i + a - \alpha a, a'), |\alpha' - \alpha|^p v(b_i, a)) \\ &\leq \max(\{|\alpha|^p + m\} \varepsilon_i, m_i v(a, b_i)) \\ &\leq \max(\varepsilon_i, \varepsilon_{i'}) \\ &= \varepsilon_i, \quad i = 1, 2, \dots, n. \end{aligned}$$

This implies that $\alpha' a' \in W_{\Sigma}(\alpha a) \subseteq U$. Thus the mapping $(\alpha, a) \rightarrow \alpha a$ is continuous.

In the last, we show that a fundamental system of neighborhoods $W_{\Sigma}(0)$ of 0 consists of only K -convex sets. For $\Sigma = \{(b_1, \varepsilon_1), (b_2, \varepsilon_2), \dots, (b_n, \varepsilon_n)\}$, suppose the points a and b belonging to the set $W_{\Sigma}(0)$.

For $i = 1, 2, \dots, n$ and $\alpha \in K$, $|\alpha| \leq 1$, we have

$$\begin{aligned} \sigma(0, b_i, \alpha a + (1 - \alpha)b) &= v(b_i, \alpha a + (1 - \alpha)b) \\ &\leq \max(|\alpha|^p v(0, b_i, a), |1 - \alpha|^p \sigma(0, b_i, b)) \\ &< \varepsilon_i. \end{aligned}$$

This implies that $\alpha a + (1 - \alpha)b \in W_{\Sigma}(0)$. Hence $W_{\Sigma}(0)$ is K-convex.

This completes the proof of the theorem.

Theorem 4. *Every non-archimedean quasi 2-normed space is a ultra 2-metric space.*

Proof.

1. On the basis of the assumption that the space has a $\dim \neq 1$, there always exists a point c to two arbitrary points a, b that are different from one another so that the vectors $b - a$ and $c - a$ are linearly independent, that is $\sigma(a, b, c) = \upsilon(b - a, c - a) \neq 0$. Further $\sigma(a, b, c) = \upsilon(b - a, c - a) = 0$ is true if at least two of the three points a, b and c are equal.

2. For arbitrary points a, b and c of X ,

$$\upsilon(b - a, c - a) = \upsilon(c - a, b - a) \quad \cdots (1).$$

Further

$$\begin{aligned} \upsilon(c - a, b - a) &= \upsilon((c - b) - (a - b), -(a - b)) \\ &= \upsilon(c - b, a - b) \quad \cdots (2). \end{aligned}$$

From the equation (1) and (2) we get

$$\sigma(a, b, c) = \sigma(a, c, b) = \sigma(b, c, a).$$

3. It is true finally

$$\begin{aligned} \sigma(a, b, c) &= \upsilon(b - a, c - a) \\ &= \upsilon((b - d) - (a - d), (c - d) - (a - d)) \\ &\leq \max(\upsilon(b - d, c - d), \upsilon(c - d, a - d), \upsilon(a - d, b - d)) \\ &= \max(\sigma(a, b, c), \sigma(a, d, c), \sigma(b, d, c)). \end{aligned}$$

This complete the proof of the theorem.

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