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On the Bordism Categories II

Elementary properties of semimodules

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In [6], it has been proved that the bordism semigroup of a bordism category is essentially reduced to the one of a semi-chain complex.

In this paper, we study elementary properties of semimodules (commutative monoids in [6]) for a homology theory of semi-chain complexes. In §4, we introduce a notion of exact sequence of semimodules, of which treatment is rather more delicate than the classical one of modules but it is enough useful for our theory. In §7 and §8, we discuss fundamental relations between semimodules and modules.

In a subsequent paper, we would introduce a notion of tensor product of semimodules and study the functors Hom and $\otimes$ for semimodules.

1. Semimodules

(1.1) A *semimodule* $A=(A,+,0)$ consists of a set A, a map $+:A\times A \to A$ and an element 0 of A such that always $x+y=y+x$, $(x+y)+z=x+(y+z)$, $x+0=x$.

A subset B of a semimodule $A=(A,+,0)$ is a *sub-semimodule* of A if $0\in B$ and if $x,y\in B$ implies $x+y\in B$.

For two semimodules $A=(A,+,0)$ and $B=(B,+,0)$, a *homomorphism* $f:A \to B$ is a map $f:A \to B$ such that always $f(0)=0$ and $f(x+y)=f(x)+f(y)$.

(1.2) A *semiring with unit* 1 $\Lambda=(\Lambda,+,0,\cdot,1)$ consists of two data:

i) $(\Lambda,+,0)$ is a semimodule, and

ii) $(\Lambda,\cdot,1)$ is a semigroup with unit 1,

such that always $(\lambda+\mu)\cdot\tau=\lambda\cdot\tau+\mu\cdot\tau$, $\lambda\cdot(\mu+\tau)=\lambda\cdot\mu+\lambda\cdot\tau$, $0\cdot\tau=\tau\cdot 0=0$.
When there is no danger of confusion, we would denote $\lambda\cdot\mu$ by $\lambda\mu$.

A subset Γ of a semiring $\Lambda=(\Lambda,+,0,\cdot,1)$ is a *sub-semiring* of Λ, if $0,1\in\Gamma$ and if $\lambda,\mu\in\Gamma$ implies $\lambda+\mu\in\Gamma$ and $\lambda\mu\in\Gamma$.

For two semirings $\Lambda=(\Lambda,+,0,\cdot,1)$ and $\Gamma=(\Gamma,+,0,\cdot,1)$, a *homomorphism* $f:\Lambda\to\Gamma$ is a map $f:\Lambda\to\Gamma$ such that always

$$f(0)=0,\ f(\lambda+\mu)=f(\lambda)+f(\mu),\ f(1)=1,\ f(\lambda\mu)=f(\lambda)f(\mu).$$

(1.3) Let $\Lambda=(\Lambda,+,0,\cdot,1)$ be a semiring. A (*left*) Λ-*semimodule* A is a semimodule $A=(A,+,0)$ together with a map $\eta:\Lambda\times A\to A$ written $\eta(\lambda,x)=\lambda x$, such that always $\lambda(x+y)=\lambda x+\lambda y$, $(\lambda+\mu)x=\lambda x+\mu x$, $(\lambda\mu)x=\lambda(\mu x)$, $1x=x$, $\lambda 0=0x=0$.

A subset B of a Λ-semimodule A is a *sub-Λ-semimodule* of A, if $x,y\in B$ implies $x+y\in B$ and if $\lambda\in\Lambda$, $x\in B$ implies $\lambda x\in B$.

For two Λ-semimodules A and B, a *Λ-homomorphism* $f:A\to B$ is a map $f:A\to B$ such that always $f(x+y)=f(x)+f(y)$ and $f(\lambda x)=\lambda f(x)$.

The set $\mathrm{Hom}_{\Lambda}(A,B)=\{f\,|\,f:A\to B\}$ of all Λ-homomorphisms of A into B is a semimodule under the addition defined for $f,g:A\to B$ by $(f+g)(x)=f(x)+g(x)$.

(1.4) For a Λ-homomorphism $f:A\to B$, if $f(a)=f(a')$ implies $a=a'$, we say that f is a *monomorphism*, while if $b\in B$ implies $b=f(a)$ for some a in A, we say that f is an *epimorphism*. Finally f is an *isomorphism* if and only if f is both a monomorphism and an epimorphism, we write $f:A\cong B$.

The following Proposition is well-known:

Proposition.

1) $f:A \to B$ is a monomorphism if and only if $f\circ g=f\circ g'$ implies $g=g'$ for all Λ-semimodule X and for all $g,g' \in \mathrm{Hom}_{\Lambda}(X,A)$.

2) $f:A \to B$ is an epimorphism if and only if $g\circ f=g'\circ f$ implies $g=g'$ for all Λ-semimodule X and for all $g,g' \in \mathrm{Hom}_{\Lambda}(B,X)$.

(1.5) For a given Λ-homomorphism $f:A \to B$, if there is a Λ-homomorphism $f':B \to A$ such that $f\circ f'\circ f=f$, we say that f is a *split Λ-homomorphism.*

Proposition. *Let $f:A \to B$ be a Λ-homomorphism. Then,*

1) $f:A \to B$ is a split monomorphism if and only if there is a Λ-homomorphism $f':B \to A$ such that $f'\circ f=1_A$.

2) $f:A \to B$ is a split epimorphism if and only if there is a Λ-homomorphism $f':B \to A$ such that $f\circ f'=1_B$.

2. Relations

(2.1) Let A be a semimodule. A *relation* ρ on A is any subset of $A\times A$. We denote the fact that an element (x,y) in $A\times A$ is in relation ρ on A by writting $x\rho y$. A relation ρ on A is called an *equivalence relation* if it satisfies the following conditions:

1) $x\rho x$, 2) $x\rho y$ implies $y\rho x$, 3) $x\rho y$, $y\rho z$ implies $x\rho z$.

An equivalence relation ρ on A is a *congruence* (*relation*) if it satisfies the following condition :

4) $x\rho y$, $z\in A$ implies $(x+z)\rho(y+z)$.

(2.2) For a given Λ-semimodule A, if we define addition and an operation of Λ on A by $(x,y)+(x',y')=(x+x',y+y')$ and $\lambda(x,y)=(\lambda x,\lambda y)$, then $A\times A$ is a Λ-semimodule.

A subset R of $A\times A$ is a *symmetric sub-semimodule* of $A\times A$ if it satisfies the following conditions:

1) R is a sub-Λ-semimodule of $A\times A$,

2) R is symmetric, i.e. $(a,b)\in R$ implies $(b,a)\in R$.

Given a symmetric sub-semimodule R of $A\times A$, we often denote (a,b) in R by $\langle a,b\rangle$. We define a relation $\rho(R)$ as follows:

$x\rho(R)y$ if and only if there exists $\langle a,b\rangle\in R$ such that $x+a= y+b$.

Proposition. *$\rho(R)$ is a congruence on A.*

$\rho(R)$ is called a *congruence associated with the symmetric sub-Λ-semimodule R of $A\times A$.*

Proof. Since $x+0=x+0$ for all x in A and $\langle 0,0\rangle\in R$, we have $x\rho(R)x$.

If $x\rho(R)y$, then $x+a=y+b$ for some $\langle a,b\rangle\in R$. Thus $y+b=x+a$ and $\langle b,a\rangle\in R$ by 2). This shows $y\rho(R)x$.

Suppose $x\rho(R)y$ and $y\rho(R)z$. Then $x+a=y+b$ and $y+a'=z+b'$ for some $\langle a,b\rangle$ and $\langle a',b'\rangle$ in R. It follows that $x+a+a'=y+b+a'=y+a'+b=z+b'+b=z+b+b'$. Since $\langle a+a',b+b'\rangle\in R$ by 1), we have $x\rho(R)z$.

Finally suppose $x\rho(R)y$ and $z\in A$. Then $x+a=y+b$ for some $\langle a,b\rangle\in R$. It follows that $x+a+z=y+b+z$, i.e. $x+z+a=y+z+b$. Since $\langle a,b\rangle\in R$, we have $(x+z)\rho(R)(y+z)$.

(2.3) $A/\rho(R)$ denotes the set of equivalence classes of A with respect to $\rho(R)$. The map $p:A\to A/\rho(R)$ that carries x in A into the equivalence class of x denoted by $[x]$ is called the *canonical surjection.* If we define addition and an operation of Λ on $A/\rho(R)$ by $[x]+[y]=[x+y]$ and $\lambda[x]=[\lambda x]$, then $A/\rho(R)$ is a Λ-semimodule and p is a Λ-homomorphism.

In particular, if B is a sub-Λ-semimodule of A, then $R=B\times B$ is a sym-

metric sub-Λ-semimodule of $A\times A$. In this case we denote $A/\rho(R)$ by A/B.

3. Kernels and images

(3.1) Let Λ be a semiring with 1. For two Λ-semimodules A and B, consider a Λ-homomorphism $f:A \to B$. Then we define $\mathrm{Ker}(f)$, $\mathrm{Im}(f)$ and $f(A)$ as follows:

$$\mathrm{Ker}(f) = \{\ a\in A \mid f(a)=0\ \},$$

$$\mathrm{Im}(f) = \{\ b\in B \mid b+f(a)=f(a') \text{ for some } a,\ a' \text{ in } A\ \},$$

$$f(A) = \{\ b\in B \mid b=f(a) \text{ for some } a \text{ in } A\ \}.$$

$\mathrm{Ker}(f)$ is called the *kernel* of f, and $\mathrm{Im}(f)$ is called the *image* of f. While $f(A)$ is called the *proper image* of f. Then $\mathrm{Ker}(f)$ is a sub-Λ-semimodule of A, and both $\mathrm{Im}(f)$ and $f(A)$ are sub-Λ- semimodules of B such that

$$f(A) \subset \mathrm{Im}(f) \subset B.$$

The natural inclusions $\mathrm{Ker}(f) \to A$, $\mathrm{Im}(f) \to B$ are denoted by $\mathrm{ker}(f)$, $\mathrm{im}(f)$ respectively.

We also introduce the notions $\mathrm{Coker}(f) = B/\mathrm{Im}(f)$ for the *cokernel* of f and $\mathrm{Coim}(f) = A/\mathrm{Ker}(f)$ for the *coimage* of f. The canonical surjections $B \to \mathrm{Coker}(f)$, $A \to \mathrm{Coim}(f)$ are denoted by $\mathrm{coker}(f)$, $\mathrm{coim}(f)$ respectively.

(3.2) Proposition. *Let $f:A \to B$ be a Λ-homomorphism. Then the followings hold:*

1) *For b,b' in B, $b+x=b'+x'$ for some x, x' in $\mathrm{Im}(f)$ if and only if $b+f(a)=b'+f(a')$ for some a, a' in A.*

2) $\mathrm{Coker}(f) = B/\mathrm{Im}(f) = B/f(A)$. *We denote it by B/f.*
In particular, if A is a sub-Λ-semimodule of B and $i:A \to B$ is the inclusion, then $B/A = B/\mathrm{Im}(i) = B/i(A)$.

Proof. 1): Suppose $b+x=b'+x'$ for some x,x' in $\mathrm{Im}(f)$. Since $x,x' \in \mathrm{Im}(f)$, $x+f(a)=f(\bar{a})$ and $x'+f(a')=f(\bar{a}')$ for some $a,\bar{a},a',\bar{a}'$ in A. $b+x=b'+x'$ implies $b+x+f(a)+f(a')=b'+x'+f(a)+f(a')$. Since $b+x+f(a)+f(a') = b+f(\bar{a})+f(a')=b+f(\bar{a}+a')$ and $b'+x'+f(a)+f(a')=b'+f(\bar{a}')+f(a)=b'+f(a+\bar{a}')$, we have $b+f(\bar{a}+a')=b'+f(a+\bar{a}')$ for $\bar{a}+a',a+\bar{a}'$ in A.

Conversely suppose $b+f(a)=b'+f(a')$ for some a,a' in A. Since $f(A)\subset\mathrm{Im}(f)$, $f(a),f(a')\in\mathrm{Im}(f)$. Thus we have $b+f(a)=b'+f(a')$ for $f(a),f(a')$ in $\mathrm{Im}(f)$.

2): $[b]=[b']$ in $B/\mathrm{Im}(f)$

$\Longleftrightarrow$ $b+x=b'+x'$ for some $\langle x,x'\rangle\in\mathrm{Im}(f)\times\mathrm{Im}(f)$

$\Longleftrightarrow$ $b+f(a)=b'+f(a')$ for some a,a' in A.

$\Longleftrightarrow$ $b+f(a)=b'+f(a')$ for some $\langle f(a),f(a')\rangle\in f(A)\times f(A)$

$\Longleftrightarrow$ $[b]=[b']$ in $B/f(A)$.

(3.3) Proposition. *Let $f:A\to B$ be a Λ-homomorphism.*

If $u:Y\to A$ is a Λ-homomorphism with $f\circ u=0$, then there is exactly one Λ-homomorphism $v:Y\to\mathrm{Ker}(f)$ with $u=(\ker(f))\circ v$.

Proof. Since $u(y)\in\mathrm{Ker}(f)$, we can define $v:Y\to\mathrm{Ker}(f)$ by $v(y)=u(y)$ for all y in Y. Since $\ker(f)$ is the inclusion, $\ker(f)\circ v(y)=u(y)$.

If $\ker(f)\circ\bar{v}(y)=u(y)$ and $\ker(f)\circ v(y)=u(y)$, then $\bar{v}=v$ since $\ker(f)$ is a monomorphism.

(3.4) Proposition. *Let $f:A\to B$ be a Λ-homomorphism.*

If $u:B\to X$ is a Λ-homomorphism with $u\circ f=0$, then there is exactly one Λ-homomorphism $v:\mathrm{Coker}(f)\to X$ with $u=v\circ\mathrm{coker}(f)$.

Proof. We define $v:\mathrm{Coker}(f)\to X$ by $v([b])=u(b)$ for all $[b]$ in $\mathrm{Coker}(f)$. If $[b]=[b']$, then $b+f(a)=b'+f(a')$ for some a,a' in A.

It follows $u(b)+u\circ f(a)=u(b')+u\circ f(a')$. Since $u\circ f=0$, $u(b)=u(b')$. This shows v is well-defined. Clearly v is a Λ-homomorphism, and $v\circ\mathrm{coker}(f)(b)=v[b]=u(b)$. Finally if $u=\bar{v}\circ\mathrm{coker}(f)$ and $u=v\circ\mathrm{coker}(f)$, then $\bar{v}=v$ since $\mathrm{coker}(f)$ is an epimorphism.

4. Exact sequences

(4.1) Let $f:A\to B$ and $g:B\to C$ be Λ-homomorphisms.

The sequence $A\xrightarrow{f}B\xrightarrow{g}C$ is called a *zero sequence* if $g\circ f=0$. The sequence $A\xrightarrow{f}B\xrightarrow{g}C$ is called *exact* if $\mathrm{Im}(f)=\mathrm{Ker}(g)$.

A sequence (finite or infinite) of Λ-homomorphisms

$$\cdots\to A_{n+1}\xrightarrow{f_{n+1}}A_n\xrightarrow{f_n}A_{n-1}\to\cdots$$

is an *exact sequence* (resp. a *zero sequence*) if the sequence $A_{n+1}\xrightarrow{f_{n+1}}A_n\xrightarrow{f_n}A_{n-1}$ ia an exact sequence (resp. a zero sequence) for each n.

By the definition an exact sequence is a zero sequence, and $g\circ f=0$ if and only if $\mathrm{Im}(f)\subset\mathrm{Ker}(g)$.

(4.2) Let $f:A\to B$ be a Λ-homomorphism.

f is called *i-regular* (*image-regular*) if $f(A)=\mathrm{Im}(f)$.

f is called *k-regular* (*kernel-regular*) if $f(a)=f(a')$ implies $a+k=a'+k'$ for some k,k' in $\mathrm{Ker}(f)$. f is *regular* if it is both i-regular and k-regular.

(4.3) For a given Λ-homomorphism $f:A\to B$, we define $f_*:\mathrm{Coim}(f)\to\mathrm{Im}(f)$ by $f_*([a])=f(a)$ for all $[a]$ in $\mathrm{Coim}(f)$. Then it is easily seen that this definition is well-defined and f_* is a Λ-homomorphism, and that f_* factors through $f(A)$.

Proposition. *Let $f:A \to B$ be a Λ-homomorphism. Then the followings hold:*

1) The sequence $0 \longrightarrow \mathrm{Ker}(f) \xrightarrow{\ker(f)} A \xrightarrow{f} B \xrightarrow{\mathrm{coker}(f)} \mathrm{Coker}(f) \longrightarrow 0$ *is exact. Where* $\ker(f)$ *is a regular monomorphism and* $\mathrm{coker}(f)$ *ia a regular epimorphism.*

2) The sequence $0 \longrightarrow \mathrm{Coim}(f) \xrightarrow{f_*} \mathrm{Im}(f) \longrightarrow 0$ *is exact.*

i) *f is k-regular* $\Longleftrightarrow$ *f_* is injective* $\Longleftrightarrow$ $\mathrm{Coim}(f) \simeq f(A)$

ii) *f is i-regular* $\Longleftrightarrow$ *f_* is surjective* $\Longleftrightarrow$ $f(A)=\mathrm{Im}(f)$

iii) *f is regular* $\Longleftrightarrow$ *f_* is an isomorphism* $\Longleftrightarrow$ $\mathrm{Coim}(f) \simeq \mathrm{Im}(f)$

Proof. 1): Consider the sequence $A \xrightarrow{f} B \xrightarrow{p} B/f$ ($p=\mathrm{coker}(f)$).

Suppose $b \in \mathrm{Im}(f)$. Then $b+f(a)=f(a')$ for some a, a' in A. It follows $[b]+[f(a)]=[f(a')]$. Since $f(a)+f(0)=0+f(a)$ and $f(a')+f(0)=0+f(a')$, $[f(a)]=[0]$ and $[f(a')]=[0]$. Therefore $[b]=[0]$. This shows $b \in \mathrm{Ker}(p)$. Hence we have $\mathrm{Im}(f) \subset \mathrm{Ker}(p)$.

Conversely suppose $b \in \mathrm{Ker}(p)$. Then $[b]=[0]$. This means $b+f(a)=f(a')$ for some a, a' in A. This shows $b \in \mathrm{Im}(f)$. Hence $\mathrm{Ker}(p) \subset \mathrm{Im}(f)$.

From the aboves the sequence $A \xrightarrow{f} B \xrightarrow{p} B/f$ is exact.

Since p is surjective, it is i-regular. Now let $p(b)=p(b')$. Then $[b]=[b']$. This means $b+f(a)=b'+f(a')$ for some a, a' in A. Since $f(a), f(a') \in f(A) \subset \mathrm{Im}(f)=\mathrm{Ker}(p)$, p is k-regular. Hence p is regular.

Next consider the sequence $\mathrm{Ker}(f) \xrightarrow{q} A \xrightarrow{f} B$ ($q=\mathrm{Ker}(f)$).

Since $f \circ q(a)=f(a)=0$ for all a in $\mathrm{Ker}(f)$, $\mathrm{Im}(q) \subset \mathrm{Ker}(f)$. Now let $a \in \mathrm{Ker}(f)$. Since q is an inclusion, $q(a)=a$. Therefore $a \in \mathrm{Im}(q)$. Hence $\mathrm{Ker}(f) \subset \mathrm{Im}(q)$. This shows $\mathrm{Ker}(f) \xrightarrow{q} A \xrightarrow{f} B$ is exact.

Since q is injective, it is k-regular. We will show that q is i-regular. It is sufficent to prove $\mathrm{Im}(q) \subset q(\mathrm{Ker}(f))$.

Let $a\in \mathrm{Im}(q)$. Then $a+q(k)=q(k')$ for some k,k' in $\mathrm{Ker}(f)$. Since q is the inclusion, $a+k=k'$ for k,k' in $\mathrm{Ker}(f)$. It follows $f(a)+f(k)=f(k')$. Since $f(k)=f(k')=0$, $f(a)=0$. Therefore $a\in \mathrm{Ker}(f)=q(\mathrm{Ker}(f))$. This proves q is i-regular. Thus q is regular.

Finally since q is injective and p is surjective, the sequences $0 \to \mathrm{Ker}(f) \overset{q}{\to} A$ and $B \overset{p}{\to} B/f \to 0$ are exact.

2): Consider $f_*: A/\mathrm{Ker}(f) \to \mathrm{Im}(f) : [a] \mapsto f(a)$. If $[a]\in \mathrm{Ker}(f_*)$, then $f_*([a])=f(a)=0$, so that $a\in \mathrm{Ker}(f)$. Hence $[a]=[0]$ in $A/\mathrm{Ker}(f)$. This shows $\mathrm{Ker}(f_*)=0$. Next we will show $\mathrm{Im}(f)/f_*=0$.

$$\begin{aligned} b\in \mathrm{Im}(f_*) &\iff b+f_*[a]=f_*[a'] \text{ for some } [a],[a'] \text{ in } A/\mathrm{Ker}(f) \\ &\iff b+f(a)=f(a') \text{ for } a,a' \text{ in } A. \\ &\iff b\in \mathrm{Im}(f) \end{aligned}$$

This shows $\mathrm{Im}(f_*)=\mathrm{Im}(f)$, and so $\mathrm{Im}(f)/f_*=0$. Thus we have the exact sequence $0 \to \mathrm{Coim}(f) \overset{f_*}{\to} \mathrm{Im}(f) \to 0$.

i): f_* is injective $\iff f_*[a]=f_*[a']$ implies $[a]=[a']$

$\iff f(a)=f(a')$ implies $a+k=a'+k'$ for some k,k' in $\mathrm{Ker}(f)$

$\iff f$ is k-regular.

ii): Suppose f_* is surjective. If $b\in \mathrm{Im}(f)$, then $f_*[a]=b$ for some $[a]$ in $A/\mathrm{Ker}(f)$, i.e. $f(a)=b$ for some a in A. This shows $f(A)=\mathrm{Im}(f)$. Hence f is i-regular.

Conversely suppose f is i-regular. Let $b\in \mathrm{Im}(f)=f(A)$. Then $b=f(a)$ for some a in A. It follows that $f_*([a])=f(a)=b$. This shows f_* is surjective.

iii): The assertion follows from the assertions i) and ii).

(4.4) By Proposition (4.3) we have easily the following:

Proposition.

1) *If* $0 \to A \xrightarrow{f} B$ *is exact, then* f *is injective if and only if* f *is* k*-regular.*

2) *If* $B \xrightarrow{g} C \to 0$ *is exact, then* g *is surjective if and only if* g *is* i*-regular.*

3) *If* $0 \to A \xrightarrow{f} B \to 0$ *is exact, then* $f: A \simeq B$ *if and only if* f *is regular.*

(4.5) Proposition. *If* $A \xrightarrow{f} B \xrightarrow{g} C$ *is exact, then*

1) $\mathrm{Im}(f)=\mathrm{Ker}(g)$, $\mathrm{Coker}(f)=\mathrm{Coim}(g)$,

2) $0 \to \mathrm{Coim}(f) \xrightarrow{f_*} \mathrm{Im}(f)=\mathrm{Ker}(g) \to 0$ *and*
$0 \to \mathrm{Coker}(f)=\mathrm{Coim}(g) \xrightarrow{g_*} \mathrm{Im}(g) \to 0$ *are exact.*

The proof is easy.

(4.6) An exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is called a *regular exact sequence* if both f and g are regular.

Proposition.

1) *If* $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ *is exact, then*

$$f \text{ is regular} \iff f_*: A \simeq \mathrm{Ker}(g).$$

2) *If* $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is exact, then*

$$g \text{ is regular} \iff g_*: \mathrm{Coker}(f) \simeq C.$$

3) *If* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is exact, then*

the sequence is a regular exact sequence

$$\iff f_*: A \simeq \mathrm{Ker}(g) \text{ and } g_*: \mathrm{Coker}(f) \simeq C.$$

Proof. It follows from Propositions (4.3) and (4.5).

Lemma. *If* $A \xrightarrow{f} B \xrightarrow{g} C$ *is exact, then*

g is k-regular $\iff$ *$g(b)=g(b')$ implies $b+f(a)=b'+f(a')$ for some a,a' in A.*

Proof. Suppose g is k-regular. Then $g(b)=g(b')$ implies $b+k=b'+k'$ for some k,k' in $\mathrm{Ker}(g)$. Since $\mathrm{Ker}(g)=\mathrm{Im}(f)$, k,k' are elements of $\mathrm{Im}(f)$. Therefore $k+f(a)=f(\bar{a})$ and $k'+f(a')=f(\bar{a}')$ for some $a,\bar{a},a',\bar{a}'$ in A. Since $b+k=b'+k'$, $b+k+f(a)+f(a')=b'+k'+f(a)+f(a')$. Since $b+k+f(a)+f(a')=b+f(\bar{a})+f(a')$ and $b'+k'+f(a)+f(a')=b'+f(a)+f(\bar{a}')$, we have $b+f(\bar{a}+a')=b'+f(a+\bar{a}')$ for $\bar{a}+a',a+\bar{a}'$ in A.

Conversely suppose that $g(b)=g(b')$ implies $b+f(a)=b'+f(a')$ for some a, a' in A. Let $g(b)=g(b')$. Then $b+f(a)=b'+f(a')$ for some a,a' in A. Since $g\circ f(a)=0$ and $g\circ f(a')=0$, $f(a)$ and $f(a')$ are elements of $\mathrm{Ker}(g)$. This shows g is k-regular.

(4.7) Let A be a sub-Λ-semimodule of a Λ-semimodule B and $i:A\to B$ be the inclusion. Since i is injective, it is k-regular. But i is not always i-regular. If i is i-regular, we call A a *regular sub-Λ-semimodule* of A.

Let $\rho(R)$ be the congruence of B associated with a symmetric sub-Λ-semimodule R of $B\times B$. Since the canonical surjection $p:B\to B/\rho(R)$ is surjection, p is i-regular. But p is not always k-regular.
Since the sequence $0\to \mathrm{Ker}(p)\xrightarrow{q} B\xrightarrow{p} B/\rho(R)\to 0$ is exact and the inclusion q is regular, $\mathrm{Ker}(p)$ is a regular sub-Λ-semimodule of B.

If p is k-regular, we call $B/\rho(R)$ a *regular quotient* of B.

Given a Λ-homomorphism $f:A\to B$, the sequence $0\to \mathrm{Ker}(f)\xrightarrow{q} A\xrightarrow{f} B\xrightarrow{p} B/f\to 0$ is exact and both q and p are regular by Proposition (4.2). Therefore $\mathrm{Ker}(f)$ is a regular sub-Λ-semimodule of A, and B/f is a regular quotient of B.

Now let A be a sub-Λ-semimodule of B, and let $i:A \to B$ be the inclusion. Then we have following exact sequences:

i) $0 \to \mathrm{Ker}(p) \overset{q}{\to} B \overset{p}{\to} B/A \to 0$

ii) $0 \to A \overset{i}{\to} B \overset{p}{\to} B/A \to 0$

iii) $0 \to A \to \mathrm{Ker}(p)=\mathrm{Im}(i) \to 0$

Where, the sequence i) is a regular exact sequence, but the sequences ii) and iii) are not always regular exact sequences. The exact sequences ii) and iii) are regular if and only if A is regular in B.

To summarize, we have proved:

Proposition.

1) $0 \to \mathrm{Ker}(p) \overset{q}{\to} B \overset{p}{\to} B/\rho(R) \to 0$ *is exact, and* q *is regular and* p *is* i*-regular.*

2) $0 \to \mathrm{Ker}(f) \overset{q}{\to} A \overset{f}{\to} B \overset{p}{\to} B/f \to 0$ *is exact, and both* q *and* p *are regular.*

3) If A *is a sub-*Λ*-semimodule of* B*, then*

i) $0 \to \mathrm{Ker}(p) \overset{q}{\to} B \overset{p}{\to} B/A \to 0$ *is a regular exact sequence.*

ii) $0 \to A \overset{i}{\to} B \overset{p}{\to} B/A \to 0$ *is exact, and* p *is regular.*

iii) $0 \to A \to \mathrm{Im}(i)=\mathrm{Ker}(p) \to 0$ *is exact.*

In particular, A *is regular in* B *if and only if* $A=\mathrm{Ker}(p)$.

(4.8) Proposition.

1) Let $0 \to A \overset{f}{\to} B \overset{g}{\to} C$ *be an exact sequence, and let* f *be a regular* Λ*-homomorphism.*

If $u:Y \to B$ *is a* Λ*-homomorphism with* $g \circ u=0$*, then there is exactly one* Λ*-homomorphism* $v:Y \to A$ *with* $u=f \circ v$.

2) Let $A \overset{f}{\to} B \overset{g}{\to} C \to 0$ *be an exact sequence, and let* g *be a regular*

Λ-homomorphism.

If $u:B \to X$ is a Λ-homomorphism with $u \circ f=0$, then there is exactly one Λ-homomorphism $v:C \to X$ with $u=v \circ g$.

Proof. 1): By Proposition (3.3), there is exactly one Λ-homomorphism $v':Y \to \mathrm{Ker}(g)$ with $\ker(g) \circ v'=u$. Since $\ker(g)$ is the inclusion, $v'(y)=u(y)$ for all y in Y. Since f is regular, $f_*:A \to \mathrm{Ker}(g):a \mapsto f(a)$ is an isomorphism by Proposition (4.6). Therefore $f_*^{-1}:\mathrm{Ker}(g) \to A$ is an isomorphism.

Now we define $v:Y \to A$ by $v=f_*^{-1} \circ v'$. Then $f \circ v(y)=f \circ f_*^{-1} \circ v'(y)=f_* \circ f_*^{-1} \circ v'(y)=v'(y)=u(y)$. Hence $f \circ v=u$.

If $\bar{v}:Y \to A$ is a Λ-homomorphism with $u=f \circ \bar{v}$, then $v=\bar{v}$ since f is injective.

2): By Proposition (3.4) there is exactly one Λ-homomorphism $v':B/f \to X$ with $u=v' \circ \mathrm{coker}(f)$, i.e. $v'([b])=u(b)$. Since g is regular, $g_*:B/f \to C:[b] \mapsto g(b)$ is an isomorphism by Proposition (4.6). Therefore $g_*^{-1}:C \to B/f$ is an isomorphism.

Now we define $v:C \to X$ by $v=v' \circ g_*^{-1}$. Then $v \circ g(b)=v' \circ g_*^{-1} \circ g(b)=v' \circ g_*^{-1} \circ g_*[b]=v'[b]=u(b)$. Hence $v \circ g=u$.

If $\bar{v}:C \to X$ is a Λ-homomorphism with $\bar{v} \circ g=u$, then $v=\bar{v}$ since g is surjective.

5. Free semimodules

(5.1) Let S be a subset of a Λ-semimodule A.

We denote by A_0 the set of all elements of the form $\sum_{s \in S} \lambda_s s$, where the λ_s are elements of Λ and equal to 0 except for a finite number of s.

We have easily:

Proposition. *A_0 is a sub-Λ-semimodule of A. Moreover it is the smallest sub-Λ-semimodule containing S.*

The sub-Λ-semimodule A_0 is said to be *generated* by S, and S is called a *system of generators* of A_0.

(5.2) Let A be a Λ-semimodule. A subset S of A is called *linearly independent* if $\Sigma_s \lambda_s s = \Sigma_s \mu_s s$ implies $\lambda_s = \mu_s$ for all s in S. A Λ-semimodule A is called a *free Λ-semimodule* if it has a linearly independent subset S of A, and then S is called a *basis* of A.

Proposition. *If A is a free Λ-semimodule with a basis S, then every element of A can be written uniquely in the form $\Sigma_{s\in S} \lambda_s s$.*

(5.3) Proposition. *To any set S there is a free Λ-semimodule $\Lambda[S]$ with S as a basis.*

Proof. Let $\Lambda[S]$ be the set of all elements of the form $\Sigma_s \lambda_s [s]$, $\lambda_s \in \Lambda$. We define an equivalence $\approx$ in $\Lambda[S]$ as follows:

$$\Sigma\, \lambda_s [s] \approx \Sigma\, \mu_s [s] \iff \lambda_s = \mu_s \text{ for all } s \text{ in } S.$$

If we define addition and an operation of Λ on $\Lambda[S]$ by

$$1^\circ \quad \Sigma\, \lambda_s [s] + \Sigma\, \mu_s [s] = \Sigma (\lambda_s + \mu_s)[s]$$

$$2^\circ \quad \lambda \Sigma\, \lambda_s [s] = \Sigma\, \lambda\lambda_s [s],$$

then we can prove easily from the above definitions that $\Lambda[S]$ is a free Λ-semimodule with S as a basis.

Regarding $s \in S$ as $1[s] \in \Lambda[S]$, we can consider as $S \subset \Lambda[S]$. We denote the inclusion by $\eta : S \to \Lambda[S]$. $\Lambda[S]$ is called the *free Λ-semimodule generated* by S.

(5.4) Proposition. *Let S be a set and A be a Λ-semimodule.*

If $f:S \to A$ is a map, then there is exactly one Λ-homomorphism $\psi:\Lambda[S] \to A$ with $f=\psi\circ\eta$.

Proof. Define $\psi:\Lambda[S] \to A$ by $\psi(a)=\Sigma\ \lambda_s f(s)$ for all $a=\Sigma\ \lambda_s[s]$ in $\Lambda[S]$. Then ψ is the required Λ-homomorphism.

(5.5) Proposition. *To any Λ-semimodule A, there are a free Λ-semimodule F and an epimorphism $:F \to A$.*

Proof. Since in the case $A=0$ the Proposition holds trivially, we may assume $A\neq 0$.

Let S be a system of generators of A. Where we may assume $0\notin S$. Let $\bar{S}$ be a copy of S. We define F by $F=\Lambda[\bar{S}]$, then F is a free Λ-semimodule. Define $\varepsilon:F \to A$ by $\varepsilon([\bar{s}])=s$. Then ε is a Λ-homomorphism, and it is an epimorphism by the definition.

6. Direct products and direct sums

(6.1) Let Λ be a semiring and $\{A_i\}_{i\in I}$ be a family of Λ-semimodules.

In the cartesian product $\Pi_{i\in I} A_i$ of $\{A_i\}_{i\in I}$, we define addition and an operation of Λ as follows:

1° $(a_i)+(a_i')=(a_i+a_i')$

2° $\lambda(a_i)=(\lambda a_i)$, $\lambda\in\Lambda$.

Then $\Pi_{i\in I} A_i$ forms a Λ-semimodule, called the *direct product* of $\{A_i\}$ and denoted by $\Pi_{i\in I} A_i$.

In the direct product $\Pi_i A_i$, the set of all elements whose components a_i are equal to 0 except for a finite number of i is denoted $\amalg_{i\in I} A_i$ and called the *external direct sum* of $\{A_i\}$.

Then $\coprod_i A_i$ is a sub-Λ-semimodule of $\prod_i A_i$. Each element (a_i) of $\coprod_i A_i$ is often denoted by $\Sigma_{i\in I}\, a_i$.

(6.2) Let $\{A_i\}_{i\in I}$ be a family of Λ-semimodules. The *canonical injection* assigning a_j to $(\cdots,0,a_j,0,\cdots)\in\prod A_i$ is denoted by $q_j:A_j \to \prod A_i$, and the *canonical surjection* assigning $(a_i)_{i\in I}$ to $a_j\in A_j$ is denoted by $p_j:\prod A_i \to A_j$. Then q_j is a monomorphism and p_j is an epimorphism.

We denote them by

(1) $A_j \overset{q_j}{\to} \coprod A_i \overset{p_j}{\to} A_j$.

Where both q_j and p_j are regular and satisfy

(2) $p_j\circ q_j=1_{A_j}$, $p_k\circ q_j=0$ $(k\neq j)$

Analogously for the $\coprod A_i$, we can define the *canonical injection* f_j and the *canonical surjection* g_j:

(3) $A_j \overset{f_j}{\to} \coprod A_i \overset{g_j}{\to} A_j$.

Then both f_j and g_j are regular and satisfy :

(4) $g_j\circ f_j=1_{A_j}$, $g_k\circ f_j=0$ $(k\neq j)$.

Moreover we have :

(5) $\Sigma_{j\in I}\, f_j\circ g_j(a)=a$ for all a in $\coprod A_i$.

(6.3) Proposition. *Let $\{A_i\}_{i\in I}$ be a family of Λ-semimodules. Then the followings hold :*

1) If C is a Λ-semimodule, and $\{u_j:C \to A_j\}_{j\in I}$ is a family of Λ-homomorphisms, then there exists a unique Λ-homomorphism $u:C \to \prod A_i$ such that

$u_j=p_j\circ u$ *for all j in I.*

2) If C is a Λ-semimodule, and $\{v_j:A_j \to C\}_{j\in I}$ is a family of Λ-homomorphisms, then there exists a unique Λ-homomorphism $v:\coprod A_i \to C$ such that

$v_j=v\circ f_j$ *for all j in I.*

Proof. Define $u:C \to \Pi A_i$ by $u(c)=(u_j(c))_{j\in I}\in\Pi A_i$ for all c in C, and $v:\amalg A_i \to C$ by $v(x)=\Sigma_{j\in I} v_j\circ g_j(x)$ for all x in $\amalg A_i$. Then we can see easily by the standard method that the assertions hold.

(6.4) Let A be a given Λ-semimodule and $\{A_i\}_{i\in I}$ be a family of sub-Λ-semimodules of A. We denote the inclusion by $\theta_i:A_i \to A$. By Proposition (6.3) we have a unique Λ-homomorphism $\theta:\amalg A_i \to A$ such that $\theta_i=\theta\circ f_i$. Since $\Sigma_i f_i\circ g_i=1$, we have $\theta=\Sigma_i\theta_i\circ g_i$, i.e. $\theta(x)=\Sigma_i x_i$ for all $x=(x_i)$ in $\amalg A_i$. A is called the *internal direct sum* of $\{A_i\}_{i\in I}$ if $\theta:\amalg_i A_i \to A$ is a Λ-isomorphism, and it is denoted by $\oplus_{i\in I} A_i$.

(6.5) **Proposition.** *Let A be a given Λ-semimodule and $\{A_i\}_{i\in I}$ be a family of sub-Λ-semimodules of A. Then the following are equivalent:*

1) *A is the internal direct sum of $\{A_i\}$: $A=\oplus_{i\in I} A_i$.*
2) *Every element a of A can be written in a unique way as $a=\Sigma_{i\in I} a_i$ where $a_i\in A_i$ for every $i\in I$ with almost all $a_i=0$.*
3) *Let $\theta_i:A_i \to A$ be the inclusion for all i in I. Then there exists a family $\{\tau_i:A \to A_i\}_{i\in I}$ of Λ-homomorphisms such that*
   (a) $\tau_i\circ\theta_i=1_{A_i}$, $\tau_k\circ\theta_i=0$ $(k\neq i)$,
   (b) *For each $a\in A$, $\tau_i(a)$ is equal to 0 except for a finite number of i, and $\Sigma_{i\in I}\theta_i\circ\tau_i(a)=a$.*

Proof. *1) => 2):* Let $a\in A$. Since θ is an epimorphism, there exists $\bar{a}=(a_i)\in\amalg A_i$ such that $\theta(\bar{a})=a$. Hence $a=\Sigma_i a_i$, $a_i\in A_i$.

If $a=\Sigma_i a_i=\Sigma_i b_i$, $a_i,b_i\in A_i$, then $\theta(\bar{a})=\theta(\bar{b})$ where $\bar{a}=(a_i),\bar{b}=(b_i)\in\amalg_i A_i$. Since θ is a monomorphism, we have $\bar{a}=\bar{b}$. Hence $a_i=b_i$ for all i in I.

2) => 3): Since every element $a\in A$ is written in a unique way as $a=\Sigma_i a_i$

where $a_i \in A_i$, we can define a map $\tau_i : A \to A_i$ by $\tau_i(a) = a_i$. Then it is easily checked that τ_i is a Λ-homomorphism satisfying :

(a) $\tau_i \circ \theta_i(a_i) = a_i$, $\tau_k \circ \theta_i(a_i) = 0$ $(k \neq i)$,

(b) $\Sigma_i \theta_i \circ \tau_i(a) = a$.

3) ⇒ 1): We define $\bar{\theta} : A \to \Pi_i A_i$ by $\bar{\theta}(a) = (\theta_i \circ \tau_i(a))_{i \in I}$ for all a in A. Then $\bar{\theta}$ is a Λ-homomorphism.

$$\theta \circ \bar{\theta}(a) = \theta(\theta_i \circ \tau_i(a)) = \Sigma_i \theta_i \circ \tau_i(a) = a.$$

$$\bar{\theta} \circ \theta(\bar{a}) = \bar{\theta}(\Sigma_j \theta_j \circ g_j(\bar{a})) = (\theta_i \circ \tau_i(\Sigma_j \theta_j \circ g_j(\bar{a})))_{i \in I} = (\Sigma_j \theta_i \circ \tau_i \circ \theta_j \circ g_j(\bar{a}))_{i \in I}$$

$$= (\theta_i \circ g_i(\bar{a}))_{i \in I} .$$

Where if $\bar{a} = (a_i)_{i \in I}$, then $\theta_i \circ g_i(\bar{a}) = \theta_i(a_i) = a_i$. Therefore $\bar{\theta} \circ \theta(\bar{a}) = (a_i)_{i \in I} = \bar{a}$. Hence θ is an isomorphism.

(6.6) Since every element $x \in \Lambda[S]$ is written in a unique way as $x = \Sigma_{s \in S} \lambda_s[s]$, $\lambda_s[s] \in \Lambda[S]$, we have the following :

Proposition. $\Lambda[S] = \oplus_{s \in S} \Lambda[s]$.

(6.7) Let $\{u_i : A_i \to B_i\}_{i \in I}$ be a family of Λ-homomorphisms.

If we define $\Pi u_i : \Pi_i A_i \to \Pi_i B_i$ by $\Pi u_i(a) = (u_i(a_i))_{i \in I}$ for all $a = (a_i)_{i \in I}$ in $\Pi_i A_i$, then Πu_i is a Λ-homomorphism.

Analogously we may define a Λ-homomorphism $\amalg u_i : \amalg_i A_i \to \amalg_i B_i$.

If $A = \oplus_i A_i$, $B = \oplus_i B_i$, we can define a Λ-homomorphism $\oplus u_i : A \to B$ by $\oplus u_i(a) = \Sigma_i u_i(a_i)$ for all $a = \Sigma_i a_i$ in A.

From the definitions we have easily that $A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i$ is exact for all i in I if and only if $\Pi_i A_i \xrightarrow{\Pi f_i} \Pi_i B_i \xrightarrow{\Pi g_i} \Pi_i C_i$ is exact.

We have also the same results for external direct sum and internal direct sum.

7. Cancellative semimodules and primitive semimodules

(7.1) A Λ-semimodule A is *cancellative* if $a+x=a'+x$ for a,a',x in A implies $a=a'$.

Proposition. *If A is cancellative, then we have the following :*

1) Every sub-Λ-semimodule of A is cancellative.

2) Let $\rho(R)$ be a congruence of A associated with a symmetric sub-Λ-semimodule R of $A\times A$. Then $A/\rho(R)$ is cancellative.

Proof. The assertion 1) follows from the definition.

2): Let $[a]+[x]=[a']+[x]$ in $A/\rho(R)$. Then $[a+x]=[a'+x]$, so that $a+x+m=a'+x+m'$ for some $\langle m,m'\rangle\in R$. Since A is cancellative, $a+m=a'+m'$. Hence $[a]=[a']$.

(7.2) If we define $\Delta(A)\subset A\times A$ by $\Delta(A)=\{\langle x,x\rangle \mid x\in A\}$, then $\Delta(A)$ is a symmetric sub-Λ-semimodule of $A\times A$. Let $\rho(\Delta(A))$ be the congruence of A associated with $\Delta(A)$: $a\ \rho(\Delta(A))\ a' \iff a+x=a'+x$ for some x in A.

We define a Λ-semimodule $c(A)$ by $c(A)=A/\rho(\Delta(A))$. The canonical surjection is denoted by $c:A\to c(A)$ which carries a in A into the equivalence class $[a]$ of a.

Proposition. *$c(A)$ is cancellative.*

Proof. Suppose $[a]+[x]=[a']+[x]$ in $c(A)$. Then $[a+x]=[a'+x]$. Thus $a+x+y=a'+x+y$ for some y in A. Since $\langle x+y,x+y\rangle\in\Delta(A)$, we have $[a]=[a']$. Hence $c(A)$ is cancellative.

(7.3) Consider the exact sequence $0\to \mathrm{Ker}(c) \xrightarrow{u} A \xrightarrow{c} c(A)\to 0$,where u is the inclusion. Then u is regular and c is an epimorphism.

Now we denote $\mathrm{Ker}(c)$ by $\delta(A)$:

$$\delta(A) = \{\ a\in A \mid a+x=x \text{ for some } x \text{ in } A\ \}.$$

Then $\delta(A)$ is a regular sub-Λ-semimodule of A.

A Λ-semimodule A is said to be *primitive* if $\delta(A)=A$.

For a given Λ-semimodule A, $\delta(A)$ is not always primitive. And also a sub-Λ-semimodule of a primitive Λ-semimodule is not always primitive.

Next we consider the exact sequence $0 \to \delta(A) \overset{u}{\to} A \overset{p}{\to} A/\delta(A) \to 0$, where p is the canonical surjection : $p(a)=<a>$.

By Proposition (3.4), there exists a unique Λ-homomorphism

$$\bar{c}: A/\delta(A) \to c(A) : <a> \mapsto [a] \quad \text{with} \quad c=\bar{c}\circ p.$$

If $\bar{c}(<a>)=0$, then $[a]=0$, i.e. $a+x=x$ for some x. This shows $\mathrm{Ker}(\bar{c})=0$. On the other hand since $\bar{c}$ is surjective, we have the exact sequence

$0 \to A/\delta(A) \overset{\bar{c}}{\to} c(A) \to 0$, where $\bar{c}$ is an epimorphism.

Thus we have the following :

Proposition. *Consider the diagram :*

$$\begin{array}{ccccccc} & & & & & & 0 \\ & & & & & & \downarrow \\ 0 & \to & \delta(A) & \overset{u}{\to} & A & \overset{p}{\to} & A/\delta(A) \to 0 \\ & & & & & c\searrow & \downarrow \bar{c} \nearrow \\ & & & & & & c(A) \\ & & & & & & \downarrow \\ & & & & & & 0 \end{array}$$

Then we have followings :

1) Λ-homomorphisms $p, \bar{c}$ and c are all surjective with $c=\bar{c}\circ p$.

2) The sequences

$$0 \to \delta(A) \overset{u}{\to} A \overset{p}{\to} A/\delta(A) \to 0$$

$$0 \to \delta(A) \overset{u}{\to} A \overset{c}{\to} c(A) \to 0$$

$$0 \to A/\delta(A) \overset{\bar{c}}{\to} c(A) \to 0$$

are exact, where the inclusion u is a regular monomorphism and p is a regular epimorphism.

(7.4) Proposition. *In the diagram :* $0 \to \delta(A) \overset{u}{\to} A \overset{p}{\to} A/\delta(A) \to 0$, with $c : A \to c(A)$ and $\bar{c} : A/\delta(A) \to c(A)$,

we have the followings :

1) $\delta(A)=0$ *if and only if* p *is injective.*

2) $A/\delta(A)$ *is cancellative* $\Longleftrightarrow$ $\bar{c}:A/\delta(A)\simeq c(A)$

$\Longleftrightarrow$ $\bar{c}$ *is injective* $\Longleftrightarrow$ $\bar{c}$ *is k-regular.*

3) A *is cancellative* $\Longleftrightarrow$ $c:A\simeq c(A)$

$\Longleftrightarrow$ c *is injective*

$\Longleftrightarrow$ p *and* $\bar{c}$ *are injective*

$\Longleftrightarrow$ $\delta(A)=0$ *and* c *is k-regular.*

4) A *is primitive* $\Longleftrightarrow$ $c(A)=0$.

Proof. 1): Suppose $\delta(A)=0$. Then $0 \to A \overset{p}{\to} A/\delta(A) \to 0$ is exact. Since p is k-regular, p is injective.

Conversely if p is injective, then $\delta(A)=\mathrm{Ker}(p)=0$ by Proposition (4.7)

2): Suppose $A/\delta(A)$ is cancellative. Let $\bar{c}\langle a\rangle=\bar{c}\langle a'\rangle$. Then $[a]=[a']$. Thus $a+x=a'+x$ for some x in A. Therefore $\langle a\rangle+\langle x\rangle=\langle a'\rangle+\langle x\rangle$ in $A/\delta(A)$. Since $A/\delta(A)$ is cancellative. we have $\langle a\rangle=\langle a'\rangle$. This shows $\bar{c}$ is injective.

Conversely suppose $\bar{c}$ is injective. Then $\bar{c}$ is an isomorphism. Since $c(A)$ is cancellative, $A/\delta(A)$ is cancellative.

3): Suppose A is cancellative. We define $c':c(A) \to A$ by $c'[a]=a$. If $[a]=[a']$, then $a+x=a'+x$ for some x. Since A is cancellative, $a=a'$. This shows c' is well-defined, and c' is a Λ-homomorphism. Then $c\circ c'[a]=c(a)=[a]$ and $c'\circ c(a)=c'[a]=a$. Hence c is an isomorphism.

Conversely suppose c is an isomorphism. Then we can show that A is cancellative since $c(A)$ is cancellative.

4): If $c(A)=0$, then $0 \to \delta(A) \overset{u}{\to} A \to 0$ is exact. Since u is a regular inclusion, we have $\delta(A)=A$.

Conversely if $\delta(A)=A$, then $A/\delta(A)=0$. Since $0 \to A/\delta(A) \overset{\bar{c}}{\to} c(A) \to 0$ is exact, we have $c(A)=0$.

(7.5) Given a Λ-homomorphism $f:A \to B$, we can define $\delta(f):\delta(A) \to \delta(B)$ by $\delta(f)=f|\delta(A)$, i.e. $\delta(f)(a)=f(a)$ for a in $\delta(A)$.

Let $a \in \delta(A)$. Then $a+x=x$ for some x in A. Therefore $f(a)+f(x)=f(x)$. This shows $f(a) \in \delta(B)$. Hence $\delta(f)$ is well-defined, and clearly it is a Λ-homomorphism.

When A is primitive, we have easily the following :

Proposition. *If A is primitive, then every Λ-homomorphism $f:A \to B$ factors through $\delta(B)$ as $f=u \circ \delta(f):A \overset{\delta(f)}{\to} \delta(B) \overset{u}{\to} B$.*

(7.6) Proposition. *If C is cancellative and $f:A \to C$ is a Λ-homomorphism, then there is exactly one Λ-homomorphism $\psi:c(A) \to C$ such that $f=\psi \circ c$.*

Proof. Define $\psi:c(A) \to C$ by $\psi([a])=f(a)$. If $[a]=[a']$, then $a+x=a'+x$ for some x in A. It follows $f(a)+f(x)=f(a')+f(x)$ in C. Since C is cancellative, $f(a)=f(a')$. This shows ψ is well-defined. It is easily seen that ψ is a Λ-homomorphism. Then $\psi \circ c(a)=\psi[a]=f(a)$. Hence $f=\psi \circ c$. Since c is an epimorphism, $\bar{\psi} \circ c=f$ and $\psi \circ c=f$ implies $\bar{\psi}=\psi$.

(7.7) Given a Λ-homomorphism $f:A \to B$, we define $c(f):c(A) \to c(B)$ by $c(f)([a])=[f(a)]$. Then $c(f)$ is well-defined, and it is a Λ-homomorphism.

We denote by Λ-*SMod* the category whose objects are Λ-semimodules and whose morphisms are Λ-homomorphisms.

The category Λ-CSMod is a subcategory of Λ-SMod whose objects are cancellative Λ-semimodules.

Next we define a functor $c:\Lambda\text{-SMod} \to \Lambda\text{-CSMod}$ as follows : The object function assigns each Λ-semimodule A to $c(A)$ and the morphism function assigns each Λ-homomorphism $f:A \to B$ to $c(f):c(A) \to c(B)$.

Let $U:\Lambda\text{-CSMod} \to \Lambda\text{-SMod}$ be the forgetful functor.

Then we have the following Theorem by Proposition (7.6).

Theorem. *The functor $c:\Lambda$-SMod $\to \Lambda$-CSMod is a left adjoint of the functor $U:\Lambda$-CSMod $\to \Lambda$-SMod. That is, if A, C are Λ-semimodules and C is cancellative, then there is a natural isomorphism :*

$$\mathrm{Hom}_{\Lambda}(A,C) \simeq \mathrm{Hom}_{\Lambda}(c(A),C).$$

(7.8) Proposition.

1) Let A be a primitive Λ-semimodule and $f:A \to B$ be a Λ-homomorphism. Then,

i) *f is i-regular $\Longrightarrow$ $\delta(f)$ is i-regular,*

ii) *f is k-regular $\Longleftrightarrow$ $\delta(f)$ is k-regular,*

iii) *f is regular $\Longrightarrow$ $\delta(f)$ is regular.*

2) Let B be a cancellative Λ-semimodule and $f:A \to B$ be a Λ-homomorphism, Then,

i) *f is i-regular $\Longleftrightarrow$ $c(f)$ is i-regular,*

ii) *f is k-regular $\Longrightarrow$ $c(f)$ is k-regular,*

iii) *f is regular $\Longrightarrow$ $c(f)$ is regular.*

Proof. 1),i): Let $b \in \mathrm{Im}(\delta(f))$. Then $b+\delta(f)(x)=\delta(f)(x')$ for some x, x' in $\delta(A)$, i.e. $b+f(x)=f(x')$ for some x,x' in $\delta(A) \subset A$.

Since f is i-regular, $b=f(a)$ for some a in A. Since $A=\delta(A)$, $b=\delta(f)(a)$. Hence $\delta(f)$ is i-regular.

ii): Suppose $\delta(f)$ is k-regular. Let $f(a)=f(a')$. Since $\delta(A)=A$, then $\delta(f)(a)=\delta(f)(a')$. By the assumption, $a+k=a'+k'$ for some k,k' in $\mathrm{Ker}(\delta(f))$. Since $k,k'\in\mathrm{Ker}(\delta(f))$ implies $k,k'\in\mathrm{Ker}(f)$, f is k-regular.

Conversely suppose f is k-regular. Let $\delta(f)(a)=\delta(f)(a')$. Then $f(a)=f(a')$. By the assumption, $a+k=a'+k'$ for $k,k'\in\mathrm{Ker}(f)$. Since $\delta(A)=A$, $k,k'\in\mathrm{Ker}(\delta(f))$. This shows $\delta(f)$ is k-regular.

The assertion iii) follows from the assertions i) and ii).

2),i): Suppose f is i-regular. Let $[b]\in\mathrm{Im}(c(f))$. Then $[b]+c(f)[x]=c(f)[x']$ for some x,x' in A, i.e. $[b+f(x)]=[f(x')]$. Therefore $b+f(x)+y=f(x')+y$ for some y in B. Since B is cancellative, $b+f(x)=f(x')$. By the assumption, $b=f(a)$ for some a in A. This shows $[b]=[f(a)]=c(f)[a]$. Hence $c(f)$ is i-regular.

Conversely suppose $c(f)$ is i-regular. Let $b\in\mathrm{Im}(f)$. Then $b+f(x)=f(x')$ for some x,x' in A. It follows $[b]+[f(x)]=[f(x')]$,i.e. $[b]+c(f)[x]=c(f)[x']$. By the assumption, $[b]=c(f)[a]$ for some a in A, i.e. $[b]=[f(a)]$. It follows $b+y=f(a)+y$ for some y. Since B is cancellative, $b=f(a)$. Hence f is i-regular.

ii): Suppose f is k-regular. Let $c(f)[a]=c(f)[a']$. Then $[f(a)]=[f(a')]$. Therefore $f(a)+x=f(a')+x$ for some x in B. Since B is cancellative, $f(a)=f(a')$. By the assumption, $a+k=a'+k'$ for some k,k' in $\mathrm{Ker}(f)$. It follows $[a]+[k]=[a']+[k']$. Since $[k],[k']\in\mathrm{Ker}(c(f))$, $c(f)$ is k-regular.

The assertion iii) follows from the assertions i) and ii).

(7.9) Proposition.

1) If $A \overset{f}{\to} B \overset{g}{\to} C$ *is exact and* A *is primitive, then the sequence* $\delta(A) \overset{\delta(f)}{\to} \delta(B) \overset{\delta(g)}{\to} \delta(C)$ *is exact.*

2) If $A \overset{f}{\to} B \overset{g}{\to} C$ *is exact and* C *is cancellative, then the sequence* $c(A) \overset{c(f)}{\to} c(B) \overset{c(g)}{\to} c(C)$ *is exact.*

Proof. 1): Since $\delta(g)\circ\delta(f)(a)=g\circ f(a)=0$ for all a in $\delta(A)$, $\mathrm{Im}(\delta(f))\subset \mathrm{Ker}(\delta(g))$. Let $b\in\mathrm{Ker}(\delta(g))$. Then $g(b)=\delta(g)(b)=0$. Thus $b\in\mathrm{Ker}(g)=\mathrm{Im}(f)$. Therefore $b+f(a)=f(a')$ for some a,a' in A. Since $\delta(A)=A$, $b+\delta(f)(a)=\delta(f)(a')$. This shows $b\in\mathrm{Im}(\delta(f))$. Therefore $\mathrm{Ker}(\delta(g))\subset\mathrm{Im}(\delta(f))$. Hence $\mathrm{Im}(\delta(f))=\mathrm{Ker}(\delta(g))$.
2): Since $c(g)\circ c(f)([a])=[g\circ f(a)]=0$, $\mathrm{Im}(c(f))\subset\mathrm{Ker}(c(g))$.

Now let $[b]\in\mathrm{Ker}(c(g))$. Then $[g(b)]=c(g)[b]=0$. Therefore $g(b)+z=z$ for some z in C. Since C is cancellative, $g(b)=0$. Thus $b\in\mathrm{Ker}(g)=\mathrm{Im}(f)$. This shows $b+f(x)=f(x')$ for some x,x' in A. It follows $[b]+c(f)[x]=c(f)[x']$. Therefore $[b]\in\mathrm{Im}(c(f))$. Thus $\mathrm{Ker}(c(g))\subset\mathrm{Im}(c(f))$. Hence $\mathrm{Im}(c(f))=\mathrm{Ker}(c(g))$.

8. K-groups and maximal sub-modules

(8.1) Let Λ be a semiring with $1\neq 0$ and A be a Λ-semimodule. And let $\Delta: A \to A\times A$ be the diagonal map defined by $\Delta(a)=(a,a)$ for all a in A. Then Δ is a Λ-homomorphism of A to $A\times A$. Since Δ is a monomorphism, the sequence $0 \to A \overset{\Delta}{\to} A\times A \overset{p}{\to} A\times A/\Delta \to 0$ is exact, where p is the canonical surjection.

We define a Λ-semimodule $K(A)$ by $K(A) = A\times A/\Delta$.
If we denote the equivalence class of $(a,b)\in A\times A$ by $\overline{(a,b)}$, the addition and the operation of Λ on $K(A)$ are given as follows :

1° $\overline{(a,b)}+\overline{(c,d)}=\overline{(a+c,b+d)}$,

2° $\lambda\overline{(a,b)}=\overline{(\lambda a,\lambda b)}$.

Since $(\overline{a,b})+(\overline{b,a})=(\overline{a+b,a+b})=(\overline{0,0})$, $(\overline{b,a})$ is the inverse element of $(\overline{a,b})$, which is written as $(\overline{b,a})=-(\overline{a,b})$.

$K(A)$ is an abelian group called the *K-group* of A.

The following equivalences hold :

$$(\overline{a,b})=(\overline{c,d}) \iff (a,b)+\Delta(x)=(c,d)+\Delta(y) \text{ for some } x,y \text{ in } A.$$

$$\iff a+x=c+y \text{ and } b+x=d+y \text{ for some } x,y \text{ in } A.$$

$$\iff a+d+w=b+c+w \text{ for some } w \text{ in } A.$$

(8.2) Let $\Lambda=(\Lambda,+,0,\cdot,1)$ be a semiring with 1. Then $K(\Lambda)$ is an abelian group with the addition : $(\overline{\lambda,\mu})+(\overline{\lambda',\mu'})=(\overline{\lambda+\lambda',\mu+\mu'})$.

We can define a *multiplication* of $K(\Lambda)$ by

$$(\overline{\lambda,\mu})(\overline{\lambda',\mu'})=(\overline{\lambda\lambda'+\mu\mu',\lambda\mu'+\mu\lambda'}).$$

Then we have the following :

Proposition.

1) If Λ is a semiring, then $K(\Lambda)$ is a ring.

2) If Λ is a semiring with 1, then

i) $1\in\delta(\Lambda) \iff K(\Lambda)=0$,

ii) $1\notin\delta(\Lambda) \iff K(\Lambda)$ *is a ring with* $1\neq 0$.

Proof. The assertion 1) is directly proved by checking the axioms of ring.

2): Suppose $1\in\delta(\Lambda)$. Then $1+x=x$ for some x in Λ. It follows $(\overline{1,0})=(\overline{1+x,x})=(\overline{0,0})$. Let $(\overline{\lambda,\mu})\in K(\Lambda)$. Then $(\overline{\lambda,\mu})=(\overline{1,0})(\overline{\lambda,\mu})=(\overline{0,0})(\overline{\lambda,\mu})=(\overline{0,0})$. This shows $K(\Lambda)=0$.

Conversely suppose $K(\Lambda)=0$. Then $(\overline{1,0})=(\overline{0,0})$. Therefore $1+x=x$ for some x in Λ. This shows $1\in\delta(\Lambda)$.

Since $\overline{(1,0)}=\overline{(0,0)}$ if and only if $1\in\delta(\Lambda)$, $\overline{(1,0)}\neq\overline{(0,0)}$ if and only if $1\notin\delta(\Lambda)$.

(8.3) Let Λ be a semiring with $1\neq0$ and A be a Λ-semimodule. Then $K(\Lambda)$ is a ring.

We can define the *operation* of $K(\Lambda)$ on $K(A)$ by $\overline{(\lambda,\mu)}\,\overline{(a,b)}=\overline{(\lambda a+\mu b,\mu a+\lambda b)}$ for $\overline{(\lambda,\mu)}\in K(\Lambda)$ and $\overline{(a,b)}\in K(A)$.

Then we have the following :

Proposition. *Let Λ be a semiring with $1\neq0$ and A be a Λ-semimodule. Then,*

1) $1\in\delta(\Lambda)\Longrightarrow K(A)=0$,

2) $1\notin\delta(\Lambda)\Longrightarrow K(A)$ *is a* $K(\Lambda)$*-module.*

Proof. 1): $1\in\delta(\Lambda)$ implies $1+\lambda=\lambda$ for some λ in Λ. Let $a\in A$. Then $(1+\lambda)a=\lambda a$, i.e. $a+\lambda a=\lambda a$. Therefore $a\in\delta(A)$. This shows A is primitive. It follows $K(A)=0$.

2): $1\notin\delta(\Lambda)$ implies $K(\Lambda)$ is a ring with $\overline{(1,0)}\neq\overline{(0,0)}$, and $\overline{(1,0)}\,\overline{(a,b)}=\overline{(a,b)}$. By checking the axioms of $K(\Lambda)$-module, it is not difficult to show that $K(A)$ is a $K(\Lambda)$-module.

(8.4) For a given Λ-homomorphism $f:A\to B$, we define a Λ-homomorphism $K(f):K(A)\to K(B)$ by $K(f)\,\overline{(a,b)}=\overline{(f(a),f(b))}$. Then $K(f)$ is well-defined and $K(f)$ is a $K(\Lambda)$-homomorphism. Since $K(f)$ is a homomorphism of $K(\Lambda)$-modules, the Λ-homomorphism $K(f)$ is always regular.

(8.5) Let Λ be a semiring with $1\neq0$ and A be a Λ-semimodule. Then we define a Λ-homomorphism $k:A\to K(A)$ by $k(a)=\overline{(a,0)}$ for all a in A.

If we define $\bar{k}:c(A) \to K(A)$ by $\bar{k}[a]=\overline{(a,0)}$, k factors uniquely through $c:A \to c(A)$ as $k=\bar{k}\circ c$. Since c is an epimorphism and $\bar{k}$ is a monomorphism, we have a Λ-isomorphism : $\bar{k}:c(A)\simeq k(A)\subset K(A)$.

Proposition. *Consider the diagram :*

$$\begin{array}{ccccccc} 0 & \to & \delta(A) & \overset{u}{\to} & A & \overset{c}{\to} & c(A) \to 0 \\ & & & & & k\searrow & \downarrow \bar{k} \nearrow \\ & & & & & & K(A) \\ & & & & & & \downarrow \\ & & & & & & 0 \end{array}$$

Then,

1) $k=\bar{k}\circ c$, *where* c *is an epimorphism and* $\bar{k}$ *is a monomorphism,*

2) *the sequence* $0 \to \delta(A) \overset{u}{\to} A \overset{k}{\to} K(A) \to 0$ *is exact,*

3) *the sequence* $0 \to c(A) \overset{\bar{k}}{\to} K(A) \to 0$ *is exact.*

Proof. The assertion 1) has been proved.

2): Consider the exact sequence $0 \to \mathrm{Ker}(k) \overset{q}{\to} A \overset{k}{\to} K(A) \overset{p}{\to} K(A)/k \to 0$. Since $a\in\mathrm{Ker}(k) \iff \overline{(a,0)}=\overline{(0,0)} \iff a+w=w$ for some $w \iff a\in\delta(A)$, we have $\mathrm{Ker}(k)=\delta(A)$ and $q=u$.

Let $\overline{(a,b)}\in K(A)$. Since $\overline{(a,b)}+\overline{(b,0)}=\overline{(a+b,b)}=\overline{(a,0)}$, then we have $\overline{(a,b)}+k(b)=k(a)$. Therefore $\overline{(a,b)}\in\mathrm{Im}(k)$, and so $K(A)\subset\mathrm{Im}(k)$. This shows $\mathrm{Im}(k)=K(A)$. Hence $K(A)/k=0$.

3): Since $\bar{k}$ is a monomorphism, $0 \to c(A) \overset{\bar{k}}{\to} K(A)$ is exact.

Now let $\overline{(a,b)}\in K(A)$. Since $\overline{(a,b)}+\overline{(b,0)}=\overline{(a,0)}$, then we have $\overline{(a,b)}+\bar{k}[b]=\bar{k}[a]$. Therefore $\overline{(a,b)}\in\mathrm{Im}(\bar{k})$. This shows $K(A)\subset\mathrm{Im}(\bar{k})$, and so $\mathrm{Im}(\bar{k})=K(A)$. Hence $c(A) \overset{\bar{k}}{\to} K(A) \to 0$ is exact.

(8.6) Proposition. *Let* Λ *be a semiring with* $1\notin\delta(\Lambda)$, *and* A *be a* Λ*-semimodule. Then there is an isomorphism of* $K(\Lambda)$*-modules :* $\bar{\bar{k}}:K(c(A))\simeq K(A)$.

Proof. Define $\bar{\bar{k}}:K(c(A)) \to K(A)$ by $\bar{\bar{k}}\{(\overline{[a],[b]})\}=\overline{(a,b)}$.

Then this is well-defined and $\bar{\bar{k}}$ is a $K(\Lambda)$-homomorphism. By the definition, $\bar{\bar{k}}$ is clearly surjective.

Now let $\bar{\bar{k}}\{([a],[b])\}=\bar{\bar{k}}\{([a'],[b'])\}$. Then $(\overline{a,b})=(\overline{a',b'})$. Therefore $a+b'+w=b+a'+w$ for some w. It follows $[a]+[b']+[w]=[b]+[a']+[w]$. This shows $([\overline{a}],[\overline{b}])=([\overline{a'}],[\overline{b'}])$. Therefore $\bar{\bar{k}}$ is injective. Hence $\bar{\bar{k}}$ is a $K(\Lambda)$-isomorphism.

(8.7) Proposition. *For $k:A \to K(A)$, we have followings :*

1) *A is primitive* $\iff$ $K(A)=0$

2) *A is cancellative* $\iff$ *k is injective*

3) *$c(A)$ is a module* $\iff$ *k is surjective*

4) $c(A)\simeq k(A)$.

Proof. 1): Suppose A is primitive. Let $(\overline{a,b})\in K(A)$. Since $a+x=x$ and $b+y=y$ for some x,y in A, we have $(\overline{a,b})=(\overline{a+x+y,b+x+y})=(\overline{x+y,x+y})=(\overline{0,0})$. This shows $K(A)=0$.

Conversely suppose $K(A)=0$. Let $a\in A$. Then $(\overline{a,0})=(\overline{0,0})$. Therefore $a+x=x$ for some x. Thus $a\in\delta(A)$. This shows $\delta(A)=A$. Hence A is primitive.

2): Suppose A is cancellative. Then c is injective by Proposition (7.4). Since $\bar{k}$ is injective, $k=\bar{k}\circ c$ is also injective.

Conversely suppose k is injective. Let $a+x=a'+x$. Then $(\overline{a,0})=(\overline{a',0})$. Therefore $k(a)=k(a')$. Since k is injective, $a=a'$. This shows A is cancellative.

3): Suppose $c(A)$ is a module. Let $(\overline{a,b})\in K(A)$. Since $c(A)$ is a module, then $[b]+[b']=[0]$ for some b'. Therefore $b+b'+x=x$ for some x. Then $k(a+b')=(\overline{a+b',0})=(\overline{a+b'+b+x,b+x})=(\overline{a,b})$. This shows k is surjective.

Conversely suppose k is surjective. Since $k=\bar{k}\circ c$, then $\bar{k}$ is surjective.

Since $\overline{k}$ is always injective, $\overline{k}$ is an isomorphism : $\overline{k}:c(A)\simeq K(A)$. Hence $c(A)$ is a module.

4): The assertion has been proved.

(8.8) Proposition. *Let* $A \overset{f}{\to} B \overset{g}{\to} C$ *be an exact sequence. Then we have followings :*

1) If B is cancellative, then

g *is k-regular* $\Longleftarrow$ $K(A) \overset{K(f)}{\to} K(B) \overset{K(g)}{\to} K(C)$ *is exact.*

2) If C is cancellative, then

g *is k-regular* $\Longrightarrow$ $K(A) \overset{K(f)}{\to} K(B) \overset{K(g)}{\to} K(C)$ *is exact.*

3) If both B and C are cancellative, then

g *is k-regular* $\Longleftrightarrow$ $K(A) \overset{K(f)}{\to} K(B) \overset{K(g)}{\to} K(C)$ *is exact.*

Proof. 1): By Lemma (4.6) it is sufficent to prove that

$g(b)=g(b') \Longrightarrow b+f(a)=b'+f(a')$ for some a,a' in A.

Let $g(b)=g(b')$. Then $K(g)(\overline{b,b'})=(\overline{g(b),g(b')})=(\overline{0,0})$. Thus $(\overline{b,b'})\in\mathrm{Ker}(K(g))$. Since $\mathrm{Ker}(K(g))\subset\mathrm{Im}(K(f))$ holds by the hypothesis, $(\overline{b,b'})\in\mathrm{Im}(K(f))$. Since $K(f)$ is regular, $\mathrm{Im}(K(f))=K(f)(K(A))$. Therefore $(\overline{b,b'})=K(f)(\overline{a,a'})$ for some $\overline{(a,a')}$ in $K(A)$, and so $(\overline{f(a),f(a')})=(\overline{b,b'})$. This means $b+f(a')+w=b'+f(a)+w$ for some w. Since B is cancellative, $b+f(a')=b'+f(a)$. Hence g is k-regular.

2): Since $K(g)\circ K(f)(\overline{a,a'})=(\overline{g\circ f(a),g\circ f(a')})=(\overline{0,0})$, $\mathrm{Im}(K(f))\subset\mathrm{Ker}(K(g))$.

Now let $(\overline{b,b'})\in\mathrm{Ker}(K(g))$. Then $K(g)(\overline{b,b'})=(\overline{0,0})$, i.e. $(\overline{g(b),g(b')})=(\overline{0,0})$. This means $g(b)+w=g(b')+w$ for some w. Since C is cancellative, $g(b)=g(b')$. Since g is k-regular, $b+f(a)=b'+f(a')$ for some a,a'. Then $K(f)(\overline{a',a})=(\overline{f(a'),f(a)})=(\overline{f(a')+b+b',f(a)+b+b'})=(\overline{b,b'})$. This shows $\mathrm{Ker}(K(g))\subset\mathrm{Im}(K(f))$. Hence $\mathrm{Im}(K(f))=\mathrm{Ker}(K(g))$.

3): The assertion follows from the assertions 1) and 2).

(8.9) Proposition. *Let* $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$ *be an exact sequence such that 1)* f *is injective, 2)* g *is regular and 3) both* B *and* C *are cancellative. Then the sequence* $0 \to K(A) \overset{K(f)}{\to} K(B) \overset{K(g)}{\to} K(C) \to 0$ *is exact.*

Proof. Since C is cancellative and g is k-regular, the sequence $K(A) \overset{K(f)}{\to} K(B) \overset{K(g)}{\to} K(C)$ is exact.

Next we will show $0 \to K(A) \overset{K(f)}{\to} K(B)$ is exact. It is sufficent to prove that $K(f)$ is injective. Let $K(f)(\overline{x,y}) = K(f)(\overline{x',y'})$. Then $(\overline{f(x),f(y)}) = (\overline{f(x'),f(y')})$. This means $f(x)+f(y')+w = f(x')+f(y)+w$ for some w. Since B is cancellative, $f(x+y') = f(x'+y)$. Since f is injective, $x+y' = x'+y$. Therefore $(\overline{x,y}) = (\overline{x',y'})$. This shows $K(f)$ is injective.

Finally we will show $K(B) \overset{K(g)}{\to} K(C) \to 0$ is exact. It is sufficent to prove that $K(g)$ is surjective. Let $(\overline{x,y}) \in K(C)$. Since g is surjective, $x = g(x')$ and $y = g(y')$ for some x', y'. It follows $K(g)(\overline{x',y'}) = (\overline{g(x'),g(y')}) = (\overline{x,y})$. This shows $K(g)$ is surjective.

Corollary. *For* $f: A \to B$ *and* $K(f): K(A) \to K(B)$ *, the followings hold :*

1) f *is surjective* $\Longrightarrow$ $K(f)$ *is surjective.*

2) f *is injective and* B *is cancellative* $\Longrightarrow$ $K(f)$ *is injective.*

3) $0 \to A \overset{f}{\to} B \to 0$ *is exact,* f *is injective and* B *is cancellative* $\Longrightarrow$ $K(A) \simeq K(B)$.

Proof. The assertions 1) and 2) have been proved in the proof of Proposition (8.9). The assertion 3) follows immediately from Proposition (8.9).

(8.10) Proposition. *Let* Λ *be a semiring with* $1 \notin \delta(\Lambda)$. *Then the followings hold :*

1) If $\{A_i\}_{i\in I}$ is a family of Λ-semimodules, then there exist $K(\Lambda)$-isomorphisms : $\Pi_i K(A_i)\simeq K(\Pi_i A_i)$, $\amalg_i K(A_i)\simeq K(\amalg_i A_i)$.

2) If $\{A_i\}_{i\in I}$ is a family of sub-Λ-semimodules of A and $A=\oplus_i A_i$, then $\oplus_{i\in I} K(A_i)=K(A)$.

Proof. 1): Define $\alpha:\Pi_i K(A_i)\to K(\Pi_i A_i)$ *by* $\alpha\{((\overline{a_i,b_i}))_{i\in I}\} = (\overline{(a_i)_{i\in I},(b_i)_{i\in I}})$. Then the assertion 1) follows easily.

2): Since every $(\overline{a,b})$ in $K(A)$ is uniquely written as $(\overline{a,b})=\Sigma_i(\overline{a_i,b_i})$ with $(\overline{a_i,b_i})\in K(A_i)$, $K(A)=\oplus_{i\in I} K(A_i)$.

Corollary. $K(\Lambda[S]) = \oplus_{s\in S} K(\Lambda[s])$.

(8.11) Let Λ be a semiring with $1\neq 0$.

A Λ-semimodule A is called a *Λ-module* if A is a module.

Let $1\notin\delta(\Lambda)$ and A be a Λ-module. Then we can define an operation of $K(\Lambda)$ on A by $(\overline{\lambda,\mu})a=\lambda a-\mu a$ for all $(\overline{\lambda,\mu})\in K(\Lambda)$ and $a\in A$. Then it is easily seen that A is a $K(\Lambda)$-module with the above definition, and that $k:A\to K(A)$ is a homomorphism of $K(\Lambda)$-modules. Since A is a module, $\delta(A)=0$. Therefore $0\to A\xrightarrow{k} K(A)\to 0$ is exact. Since both A and $K(A)$ are modules, k is regular. Thus $k:A\simeq K(A)$. Since k is a $K(\Lambda)$-homomorphism, k is an isomorphism of $K(\Lambda)$-modules.

Conversely if A is a $K(\Lambda)$-module, then A is necessarily a Λ-module.

Thus we have :

Proposition. *Let Λ be a semiring with $1\notin\delta(\Lambda)$. Then the followings hold:*

1) A is a Λ-module if and only if A is a $K(\Lambda)$-module.

2) If A is a Λ-module, then $k:A\to K(A)$ gives an isomorphism of $K(\Lambda)$-modules.

(8.12) Let A be a Λ-semimodule. Define a sub-Λ-semimodule $G(A)$ by

$$G(A) = \{\ a\in A \mid a+a'=0 \text{ for some } a' \text{ in } A\ \}.$$

Then $G(A)$ is a module by the definition, and it is a maximal module contained in A. Thus $G(A)$ is a Λ-module called the *maximal sub-module* of A. Consider the exact sequence $0 \to G(A) \overset{v}{\to} A \overset{p}{\to} A/G(A) \to 0$, where v is the inclusion and p is the canonical surjection. It is easily seen that v is regular. Therefore $G(A)$ is a regular sub-Λ-semimodule of A.

By Proposition (8.11), we have the following :

Proposition. *If Λ is a semiring with $1\notin\delta(\Lambda)$, then $G(A)$ is a $K(\Lambda)$-module.*

(8.13) Given a Λ-homomorphism $f:A \to B$, we define $G(f):G(A) \to G(B)$ by $G(f)(a)=f(a)$. Then $G(f)$ is well-defined and it is always regular.

Proposition. *If $A \overset{f}{\to} B \overset{g}{\to} C$ is exact and A is a Λ-module, then the sequence $G(A) \overset{G(f)}{\to} G(B) \overset{G(g)}{\to} G(C)$ is exact.*

Proof. Since $G(g)\circ G(f)(a)=g\circ f(a)=0$, $\mathrm{Im}(G(f))\subset\mathrm{Ker}(G(g))$.

Now let $b\in\mathrm{Ker}(G(g))$. Then $g(b)=G(g)(b)=0$. Thus $b\in\mathrm{Ker}(g)=\mathrm{Im}(f)$. This means $b+f(a)=f(\bar{a})$ for some $a,\bar{a}$ in A. Since $G(A)=A$, $b+G(f)(a)=G(f)(\bar{a})$. Therefore $b\in\mathrm{Im}(G(f))$. This shows $\mathrm{Ker}(G(g))\subset\mathrm{Im}(G(f))$. Hence $\mathrm{Im}(G(f))=\mathrm{Ker}(G(g))$.

(8.14) Proposition. *If R is a ring with $1\neq 0$, then $k:R \to K(R)$ gives an isomorphism of rings.*

Proof. By the hypothesis $\delta(R)=0$, and so $1\notin\delta(R)$. By Propositon (8.11), k is an isomorphism of modules. Since $k(\lambda\mu)=(\overline{\lambda\mu,0})=(\overline{\lambda,0})(\overline{\mu,0})=k(\lambda)k(\mu)$,

k is an isomorphism of rings.

(8.15) Proposition. *If R is a ring with $1\neq 0$ and A is a R-module, Then $K(A)$ is a R-module and $k:A \to K(A)$ gives an isomorphism of R-modules.*

Proof. Since $K(A)$ is a $K(R)$-module and $k:A \to K(A)$ is an isomorphism of $K(R)$-modules, $K(A)$ is necessarily R-module and k is necessarily an isomorphism of R-modules with the operation of R in $K(A)$ defined by $r(\overline{a,b})=(\overline{ra,rb})$ for $r\in R$ and $(\overline{a,b})\in K(A)$.

(8.16) Let Λ be a semiring with $1\notin s(\Lambda)$. If G is a $K(\Lambda)$-module, then G is naturaly a Λ-semimodule by $\lambda a=(\overline{\lambda,0})a$ for $\lambda\in\Lambda$ and $a\in G$, which is denoted by $U(G)$. If $\psi:G \to G'$ is a $K(\Lambda)$-homomorphism, then it is necessarily a Λ-homomorphism denoted by $U(\psi):U(G) \to U(G')$.

Proposition. *Let Λ be a semiring with $1\notin s(\Lambda)$. Then we have the following universal properties :*

1) If $f:G \to A$ is a Λ-homomorphism and G is a Λ-module, then there exists exactly one $K(\Lambda)$-homomorphism $\psi:G \to G(A)$ such that $f=v\circ U(\psi)$, where v is the inclusion $v:G(A) \to A$. In this case ψ is given by $G(f)$.

2) If $f:A \to G$ is a Λ-homomorphism and G is a Λ-module, then there exists exactly one $K(\Lambda)$-homomorphism $\psi:K(A) \to G$ such that $f=U(\psi)\circ k$, where $k:A \to K(A)$.

Proof. 1): Define $\psi:G \to G(A)$ by $\psi(x)=f(x)$ for all x in G. Since $G=G(G)$, ψ is given by $G(f)$. Then clearly $f=v\circ U(\psi)$. If $f=v\circ U(\psi)=v\circ U(\overline{\psi})$, then $U(\psi)=U(\overline{\psi})$ since v is injective, and so $\psi=\overline{\psi}$.

2): Define $\psi:K(A) \to G$ by $\psi(\overline{a,b})=f(a)-f(b)$. If $(\overline{a,b})=(\overline{a',b'})$, then $a+b'+w=b+a'+w$ for some w. Therefore $f(a)+f(b')+f(w)=f(b)+f(a')+f(w)$.

Since G is a mudule, $f(a)-f(b)=f(a')-f(b')$. This shows ψ is well-defined. Clearly ψ is a $K(\Lambda)$-homomorphism. Then $U(\psi)\circ k(a)=U(\psi)(\overline{a,0})=\psi(\overline{a,0})=f(a)$ for all a in A.

Finally if $f=U(\bar{\psi})\circ k$, then $\bar{\psi}(\overline{a,0})=f(a)$ for a in A, and so $\bar{\psi}(\overline{a,b})=\bar{\psi}\{(\overline{a,0})-(\overline{b,0})\}=\bar{\psi}(\overline{a,0})-\bar{\psi}(\overline{b,0})=f(a)-f(b)=\psi(\overline{a,b})$. Hence $\psi=\bar{\psi}$.

(8.17) Let Λ be a semiring with $1\notin\delta(\Lambda)$.

We denote by $K(\Lambda)$-*Mod* the category whose objects are $K(\Lambda)$-modules and whose morphisms are $K(\Lambda)$-homomorphisms.

We define a functor $G:\Lambda$-*SMod* $\to K(\Lambda)$-*Mod* as follows: The object function assigns each Λ-semimodule A to $G(A)$ and the morphism function assigns each Λ-homomorphism $f:A\to B$ to $G(f):G(A)\to G(B)$.

Next we define a functor $K:\Lambda$-*SMod* $\to K(\Lambda)$-*Mod* as follows: The object function assigns each Λ-semimodule A to $K(A)$ and the morphism function assigns each Λ-homomorphism $f:A\to B$ to $K(f):K(A)\to K(B)$.

Finally let $U:K(\Lambda)$-*Mod* $\to\Lambda$-*SMod* be the forgetful functor.

Then we have the following Theorem by Proposition (8.16).

Theorem. *Let Λ be a semiring with $1\notin\delta(\Lambda)$. Then the followings hold :*

1) The functor $G:\Lambda$-SMod $\to K(\Lambda)$-Mod is a right adjoint of the forgetful functor $U:K(\Lambda)$-Mod $\to\Lambda$-SMod. That is, if G is a $K(\Lambda)$-module and A is a Λ-semimodule, then there is a natural isomorphism :

$$\mathrm{Hom}_{\Lambda}(U(G),A)\cong\mathrm{Hom}_{K(\Lambda)}(G,G(A)).$$

2) The functor $K:\Lambda$-SMod $\to K(\Lambda)$-Mod is a left adjoint of the forgetful functor $U:K(\Lambda)$-Mod $\to\Lambda$-SMod. That is, if A is a Λ-semimodule and G is a $K(\Lambda)$-module, then there is a natural isomorphism :

$$\mathrm{Hom}_{\Lambda}(A,U(G))\cong\mathrm{Hom}_{K(\Lambda)}(K(A),G).$$

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